On First and Second Order Planar Elliptic Equations with Degeneracies

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Abstract. This paper deals with elliptic equations in the plane with degeneracies. The equations are generated by a complex vector field that is elliptic everywhere except along a simple closed curve. Kernels for these equations are constructed. Properties of solutions, in a neighborhood of the degeneracy curve, are obtained through integral and series representations. An application to a second order elliptic equation with a punctual singularity is given.

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CONTENTS

Introduction

This paper deals with the properties of solutions of first and second order equations in the plane. These equations are generated by a complex vector field X that is elliptic everywhere except along a simple closed curve $\Sigma \subset \mathbb{R}^2$. The vector field X is tangent to Σ and $X \wedge \overline{X}$ vanishes to first order along Σ (and so X does not satisfy Hörmander's bracket condition). Such vector fields have canonical representatives (see [7]). More precisely, there is a change of coordinates in a tubular neighborhood of Σ such that X is conjugate to a unique vector field L of the form

$$L = \lambda \frac{\partial}{\partial t} - ir \frac{\partial}{\partial r} \tag{0.1}$$

defined in a neighborhood of the circle $r = 0$ in $\mathbb{R} \times \mathbb{S}^1$, where $\lambda \in \mathbb{R}^+ + i\mathbb{R}$ is an invariant of the structure generated by X. We should point out that normalizations for vector fields X such that $X \wedge \overline{X}$ vanishes to a constant order $n > 1$ along Σ are obtained in [8], but we will consider here only the case $n = 1$. This canonical representation makes it possible to study the equations generated by X in a neighborhood of the characteristic curve Σ. We would like to mention a very recent paper by F. Treves [12] that uses this normalization to study hypoellipticity and local solvability of complex vector fields in the plane near a linear singularity. The motivation for our work stems from the theory of hypoanalytic structures (see [11] and the references therein) and from the theory of generalized analytic functions (see [17]).

The equations considered here are of the form $Lu = F(r, t, u)$. and $Pu = G(r, t, u, Lu)$, where P is the (real) second order operator

$$P = L\overline{L} + \beta(t)L + \overline{\beta}(t)\overline{L}\,. \tag{0.2}$$

It should be noted that very little is known, even locally, about the structure of the solutions of second order equations generated by complex vector fields with degeneracies. The paper [5] explores the local solvability of a particular case generated by a vector field of finite type.

An application to a class of second order elliptic operators with a punctual singularity in $\mathbb{R}^2$ is given. This class consists of operators of the form

$$D = a_{11}\frac{\partial^2}{\partial x^2} + 2a_{12}\frac{\partial^2}{\partial xy} + a_{22}\frac{\partial^2}{\partial y^2} + a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y}\,, \tag{0.3}$$

where the coefficients are real-valued, smooth, vanish at 0, and satisfy

$$C_1 \,\leq\, \frac{a_{11}a_{22} - a_{12}^2}{(x^2+y^2)^2} \,\leq\, C_2$$

for some positive constants $C_1 \leq C_2$. It turns out that each such operator D is conjugate in $U\backslash 0$ (where U is an open neighborhood of $0 \in \mathbb{R}^2$) to a multiple of an operator P given by (0.2).

Our approach is based on a thorough study of the operator $\mathcal{L}$ given by

$$\mathcal{L}u = Lu - A(t)u - B(t)\overline{u}\,. \tag{0.4}$$

For the equation $\mathcal{L}u = 0$, we introduce particular solutions, called here basic solutions. They have the form

$$w(r,t) = r^{\sigma}\phi(t) + \overline{r^{\sigma}\psi(t)}\,,$$

where $\sigma \in \mathbb{C}$ and $\phi(t)$, $\psi(t)$ are 2π-periodic and $\mathbb{C}$-valued. Sections 2 and 4 establish the main properties of the basic solutions. In particular, we show that for every $j \in \mathbb{Z}$, there are (up to real multiples) exactly two $\mathbb{R}$-independent basic solutions

$$w_j^{\pm}(r,t) = r^{\sigma_j^{\pm}}\phi_j^{\pm}(t) + \overline{r^{\sigma_j^{\pm}}\psi_j^{\pm}(t)}$$

with winding number j. For a given j, if $\sigma_j^+ \in \mathbb{C}\backslash\mathbb{R}$, then $\sigma_j^- = \sigma_j^+$; and if $\sigma_j^+ \in \mathbb{R}$ then we have only $\sigma_j^- \leq \sigma_j^+$. The basic solutions play a fundamental role in the structure of the space of solutions of the equation $\mathcal{L}u = F$.

In section 6, we show that any solution of $\mathcal{L}u = 0$ in a cylinder $(0,\ R)\times\mathbb{S}^1$ has a Laurent type series expansion in the $w_j^{\pm}$'s. From the basic solutions of $\mathcal{L}$ and those of the adjoint operator $\mathcal{L}^*$, we construct, in section 5, kernels Ω_1 and Ω_2 that allow us to obtain a Cauchy Integral Formula (section 6)

$$u(r,t) = \int_{\partial_0 U} \Omega_1 u\,\frac{d\zeta}{\zeta} + \overline{\Omega_2 u}\,\frac{d\overline{\zeta}}{\overline{\zeta}} \tag{0.5}$$

that represents the solution u of $\mathcal{L}u = 0$ in terms of its values on the distinguished boundary $\partial_0 U = \partial U\backslash\Sigma$.

For the nonhomogeneous equation $\mathcal{L}u = F$, we construct, in section 7, an integral operator T, given by

$$TF = \frac{-1}{2\pi} \iint_U \left(\Omega_1 F + \overline{\Omega_2 F}\right) \frac{d\rho d\theta}{\rho} \,. \tag{0.6}$$

This operator produces Hölder continuous solutions (up to the characteristic circle Σ), when F is in an appropriate L^p-space. The properties of T allow us to establish, in section 8, a similarity principle between the solutions of the homogeneous equations $\mathcal{L}u = 0$ and those of a semilinear equation $\mathcal{L}u = F(r, t, u)$

The properties of the (real-valued) solutions of $Pu = G$ are studied in sections 9 to 11. To each function u we associate a complex valued function $w = BLu$, called here the L-potential of u, and such that w solves an equation of the form $\mathcal{L}w = F$. The properties of the solutions of $Pu = G$ can thus be understood in terms of the properties of their L-potentials. In particular series representations and integral representations are obtained for u. A maximum principle for the solutions of $Pu = 0$ holds on the distinguished boundary $\partial_0 U$, if the spectral values $\sigma_j^{\pm}$ satisfy a certain condition. In the last section, we establish the conjugacy between the operator D and the operator P.

1 Preliminaries

We start by reducing the main equation $Lu = Au + B\overline{u}$ into a simpler form. Then, we define a family of operators $\mathcal{L}_\epsilon$, their adjoint $\mathcal{L}_\epsilon^*$, and prove a Green's formula. The operators $\mathcal{L}_\epsilon$ will be extensively used in the next section.

Let $\lambda = a + ib \in \mathbb{R}^+ + i\mathbb{R}^*$ and define the vector field L by

$$L = \lambda \frac{\partial}{\partial t} - ir \frac{\partial}{\partial r} \,. \tag{1.1}$$

For $A \in C^k(\mathbb{S}^1, \mathbb{C})$, with $k \in \mathbb{Z}^+$, set

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} A(t) dt, \qquad \nu = 1 - \mathrm{Im}\frac{A_0}{\lambda} + \left[\mathrm{Im}\frac{A_0}{\lambda}\right]$$

where for $x \in \mathbb{R}$, $[x]$ denotes the greatest integer less or equal than x. Hence, $\nu \in [0,\ 1)$. Define the function

$$m(t) = \exp\left(it + i\left[\mathrm{Im}\frac{A_0}{\lambda}\right] t + \frac{1}{\lambda}\int_0^t (A(s) - A_0) ds\right) \,.$$

Note that $m(t)$ is 2π-periodic. The following lemma is easily verified.

Lemma 1.1 *Let $A, B \in C^k(\mathbb{S}^1, \mathbb{C})$ and $m(t)$ be as above. If $u(r,t)$ is a solution of the equation*

$$Lu = A(t)u + B(t)\overline{u} \tag{1.2}$$

then the function $w(r,t) = \dfrac{u(r,t)}{m(t)}$ solves the equation

$$Lw = \lambda\left(\mathrm{Re}\frac{A_0}{\lambda} - i\nu\right)w + C(t)\overline{w} \tag{1.3}$$

where $C(t) = B(t)\dfrac{\overline{m}(t)}{m(t)}$.

In view of this lemma, from now on, we will assume that $\mathrm{Re}\dfrac{A_0}{\lambda} = 0$ and deal with the simplified equation

$$Lu = -i\lambda\nu u + c(t)\overline{u} \tag{1.4}$$

where $\nu \in [0,1)$ and $c(t) \in C^k(\mathbb{S}^1, \mathbb{C})$.

Consider the family of vector fields

$$L_\epsilon = \lambda_\epsilon\frac{\partial}{\partial t} - ir\frac{\partial}{\partial r} \tag{1.5}$$

where $\lambda_\epsilon = a + ib\epsilon,\ \ \epsilon \in \mathbb{R}$, and the associated operators $\mathcal{L}_\epsilon$ defined by

$$\mathcal{L}_\epsilon u(r,t) = \lambda_\epsilon\frac{\partial u}{\partial t}(r,t) - ir\frac{\partial u}{\partial r}(r,t) + i\lambda_\epsilon\nu u(r,t) - c(t)\overline{u(r,t)} \tag{1.6}$$

For $\mathbb{C}$-valued functions defined on an open set $U \in \mathbb{R}^+ \times \mathbb{S}^1$, we define the bilinear form

$$< f, g >= \mathrm{Re}\left(\iint_U f(r,t)g(r,t)\frac{drdt}{r}\right).$$

For the duality induced by this form, the adjoint of $\mathcal{L}_\epsilon$ is

$$\mathcal{L}_\epsilon^* v(r,t) = -\left(\lambda_\epsilon\frac{\partial v}{\partial t}(r,t) - ir\frac{\partial v}{\partial r}(r,t) - i\lambda_\epsilon\nu v(r,t) + \overline{c(t)}\,\overline{v(r,t)}\right) \tag{1.7}$$

The function $z_\epsilon(r,t) = |r|^{\lambda_\epsilon}\mathrm{e}^{it}$ is a first integral of L_ϵ in $\mathbb{R}^* \times \mathbb{S}^1$. That is, $L_\epsilon z_\epsilon = 0$, $dz_\epsilon \neq 0$. Furthermore $z_\epsilon : R^+ \times \mathbb{S}^1 \longrightarrow \mathbb{C}^*$ is a diffeomorphism. The following Green's identity will be used throughout.

Proposition 1.1 *Let $U \subset \mathbb{R}^+ \times \mathbb{S}^1$ be an open set with piecewise smooth boundary. Let $u, v \in C^0(\overline{U})$ with $L_\epsilon u$ and $L_\epsilon v$ integrable. Then,*

$$\mathrm{Re}\left(\int_{\partial U} uv\frac{dz_\epsilon}{z_\epsilon}\right) =< u, \mathcal{L}_\epsilon^* v > - < \mathcal{L}_\epsilon u, v >. \tag{1.8}$$

Proof. Note that for a differentiable function $f(r,t)$, we have

$$df = \frac{i}{2a}\left(-\overline{L_\epsilon} f \frac{dz_\epsilon}{z_\epsilon} + L_\epsilon f \frac{d\overline{z_\epsilon}}{\overline{z_\epsilon}}\right) \quad \text{and} \quad \frac{d\overline{z_\epsilon}}{\overline{z_\epsilon}} \wedge \frac{dz_\epsilon}{z_\epsilon} = \frac{2ia}{r} dr \wedge dt\,.$$

Hence,

$$\begin{aligned}\int_{\partial U} uv \frac{dz_\epsilon}{z_\epsilon} &= \iint_U \frac{i}{2a}\left(uL_\epsilon v + vL_\epsilon u\right) \frac{d\overline{z_\epsilon}}{\overline{z_\epsilon}} \wedge \frac{dz_\epsilon}{z_\epsilon} \\ &= -\iint_U \left(v\mathcal{L}_\epsilon u - u\mathcal{L}_\epsilon^* v + cv\overline{u} - u\overline{cv}\right) \frac{drdt}{r}\end{aligned}$$

By taking the real parts, we get (1.8) $\square$

Remark 1.1 When $b = 0$ so that $\lambda = a \in \mathbb{R}^+$. The pushforward via the first integral $r^a e^{it}$ reduces the equation $\mathcal{L}u = F$ into a Cauchy Riemann equation with a singular point of the form

$$\frac{\partial W}{\partial \overline{z}} = \frac{a_0}{\overline{z}} W + \frac{B(t)}{\overline{z}} \overline{W} + G(z) \tag{1.9}$$

Properties of the solutions of such equations are thoroughly studied in [9]. Many aspects of CR equations with punctual singularities have been studied by a number of authors and we would like to mention in particular the following papers [1], [13], [14], [15] and [16] .

Remark 1.2 We should point out that the vector fields involved here satisfy the Nirenberg-Treves Condition (P) at each point of the characteristic circle. For vector fields X satisfying condition (P), there is a rich history for the local solvability of the $\mathbb{C}$-linear equation $Xu = F$ (see the books [3], [11] and the references therein). Our focus here is first, on the semiglobal solvability in a tubular neighborhood of the characteristic circle, and second, on the equations containing the term $\overline{u}$ which makes them not $\mathbb{C}$-linear.

Remark 1.3 The operator $\mathcal{L}_\epsilon$ is invariant under the diffeomorphism $\Phi(r,t) = (-r,t)$ from $\mathbb{R}^+ \times \mathbb{S}^1$ to $\mathbb{R}^- \times \mathbb{S}^1$. Hence, all the results about $\mathcal{L}_\epsilon$ stated in domains contained in $\mathbb{R}^+ \times \mathbb{S}^1$ have their counterparts for domains in $\mathbb{R}^- \times \mathbb{S}^1$. Throughout this paper, we will be mainly stating results for $r \geq 0$.

2 Basic Solutions

In this section we introduce the notion of basic solutions for $\mathcal{L}_\epsilon$. We say that w is a basic solution of $\mathcal{L}_\epsilon$ if it is a nontrivial solution of $\mathcal{L}_\epsilon w = 0$, in $\mathbb{R}^+ \times \mathbb{S}^1$, of the form

$$w(r,t) = r^\sigma \phi(t) + \overline{r^\sigma \psi(t)}, \tag{2.1}$$

with $\sigma \in \mathbb{C}$ and where $\phi(t), \psi(t)$ are 2π-periodic functions. These solutions play a crucial role for the equations generated by L_ϵ. In a sense, they play roles similar to those played by the functions z^n in classical complex and harmonic analysis.

Consider, as our starting point, the basic solutions of $\mathcal{L}_0$. These basic solutions are known, since they can be recovered from those of equation (1.9) (see Remark 1.1). From $\mathcal{L}_0$, we obtain the properties of the basic solution for $\mathcal{L}_\epsilon$. This is done through continuity arguments in the study of an associated system of 2×2 ordinary differential equations in $\mathbb{C}^2$ with periodic coefficients. By using analytic dependence of the system with respect to the parameters, the spectral values σ of the monodromy matrix can be tracked down. The main result (Theorem 2.1) states that for every $j \in \mathbb{Z}$, the operator $\mathcal{L}_\epsilon$ has exactly two $\mathbb{R}$-independent basic solutions with winding number j.

2.1 Properties of basic solutions

We prove that a basic solution has no vanishing points when $r > 0$ and that one of its components ϕ or ψ is always dominating.

It is immediate, from (1.6), that in order for a function $w(r,t)$, given by (2.1), to satisfy $\mathcal{L}_\epsilon w = 0$, the components ϕ and ψ need to be periodic solutions of the system of ordinary differential equations

$$\left\{ \begin{array}{l} \lambda_\epsilon \phi'(t) = i(\sigma - \lambda_\epsilon \nu)\phi(t) + c(t)\psi(t) \\ \overline{\lambda_\epsilon}\psi'(t) = -i(\sigma - \overline{\lambda_\epsilon}\nu)\psi(t) + \overline{c(t)}\phi(t)\,. \end{array} \right. \tag{2.2}$$

Note that if $\sigma \in \mathbb{R}$, then $w = r^\sigma(\phi(t) + \overline{\psi(t)})$ and $f = \phi + \overline{\psi}$ solves the equation

$$\lambda_\epsilon f'(t) = i(\sigma - \lambda_\epsilon \nu) f(t) + c(t)\overline{f(t)}\,. \tag{2.3}$$

Now we prove that a basic solution cannot have zeros when $r > 0$.

Proposition 2.1 *Let $w(r,t)$, given by* (2.1)*, be a basic solution of $\mathcal{L}_\epsilon$. Then*

$$w(r,t) \neq 0 \qquad \forall (r,t) \in \mathbb{R}^+ \times \mathbb{S}^1\,.$$

Proof. If $\sigma \in \mathbb{R}$, we have $w(r,t) = r^\sigma f(t)$ with $f(t)$ satisfying (2.3). If $w(r_0,t_0) = 0$ for some $r_0 > 0$, then $f(t_0) = 0$ and so $f \equiv 0$ by uniqueness of solutions of the differential equation (2.3). Now, assume that $\sigma = \alpha + i\beta$ with $\beta \in \mathbb{R}^*$. Suppose that w is a basic solution and $w(r_0,t_0) = 0$ for some $(r_0,t_0) \in \mathbb{R}^+ \times \mathbb{S}^1$. Consider the sequence of real numbers $r_k = r_0 \exp(-k\pi/|\beta|)$ with $k \in \mathbb{Z}^+$. Then $r_k \longrightarrow 0$ as $k \longrightarrow \infty$ and $r_k^{2i\beta} = r_0^{2i\beta}$. It follows at once from $w(r_0,t_0) = 0$ and (2.1) that $w(r_k,t_0) = 0$ for every $k \in \mathbb{Z}^+$. Note that from (2.1) we have $|w(r,t)| \leq Er^a$, where

$E = \max(|\phi(t)| + |\psi(t)|)$. Note also that since $\mathcal{L}_\epsilon$ is elliptic in $\mathbb{R}^+ \times \mathbb{S}^1$, then the zeros of any solution of the equation $\mathcal{L}_\epsilon u = 0$ are isolated in $\mathbb{R}^+ \times \mathbb{S}^1$.

The pushforward via the mapping $z = r^{\lambda_\epsilon} \mathrm{e}^{it}$ of the equation $\mathcal{L}_\epsilon w = 0$ in $\mathbb{R}^+ \times \mathbb{S}^1$ is the singular CR equation

$$\frac{\partial W}{\partial \overline{z}} = \frac{\lambda_\epsilon \nu \mathrm{e}^{2i\theta}}{2az} W - \frac{C(z)\mathrm{e}^{2i\theta}}{2iaz} \overline{W}$$

where $W(z)$ and $C(z)$ are the pushforwards of $w(r,t)$ and $c(t)$ and where θ is the argument of z. We are going to show that W has the form $W(z) = H(z)\exp(S(z))$ where H is holomorphic in the punctured disc $D^*(0,R)$, $S(z)$ continuous in $D^*(0,R)$ and satisfies the growth condition $|S(z)| \leq \log \dfrac{K}{|z|^p}$ for some positive constants K and p. For this, consider the function $M(z)$ defined by

$$M(z) = \frac{\lambda_\epsilon \nu \mathrm{e}^{2i\theta}}{2a} - \frac{C(z)\mathrm{e}^{2i\theta}}{2ia} \frac{\overline{W(z)}}{W(z)}$$

for $0 < |z| < R$, $W(z) \neq 0$ and by $M(z) = 1$ on the set of isolated points where $W(z) = 0$. This function is bounded and it follows from the classical theory of CR equations (see [2] or [17]) that

$$N(z) = \frac{-1}{\pi} \iint_{D(0,R)} \frac{M(\zeta)}{\zeta - z} d\xi d\eta$$

($\zeta = \xi + i\eta$) is continuous, satisfies $\dfrac{\partial N(z)}{\partial \overline{z}} = M(z)$ and

$$|N(z_1) - N(z_2)| \leq A||M||_\infty |z_1 - z_2| \log \frac{2R}{|z_1 - z_2|} \quad \forall z_1, z_2 \in D(0,R)$$

for some positive constant A. Define S by $S(z) = \dfrac{N(z) - N(0)}{z}$. We have then, for $z \neq 0$,

$$\frac{\partial S}{\partial \overline{z}} = \frac{W_{\overline{z}}(z)}{W(z)} \quad \text{and} \quad |S(z)| \leq B \log \frac{2R}{|z|}\,.$$

with $B = A||M||_\infty$ Let $H(z) = W(z)\exp(-S(z))$. Then H is holomorphic in $0 < |z| < R$ and it satisfies

$$|H(z)| \leq |W(z)| \exp(|S(z)|) \leq |W(z)| \frac{(2R)^B}{|Z|^B} \leq C_1 |z|^s\,.$$

for some constants C_1 and $s \in R$. The last inequality follows from the estimate $|w| \leq Er^\alpha$. This means that the function H has at most a pole at $z = 0$. Since $w(r_k, t_0) = 0$, then $H(z_k) = 0$ for every k and $z_k = r_k^{\lambda_\epsilon} \mathrm{e}^{it_0} \longrightarrow 0$. Hence $H \equiv 0$ and $w \equiv 0$ which is a contradiction $\square$

Corollary 2.0.1 *If $w = r^\sigma\phi(t) + \overline{r^\sigma\psi(t)}$ is a basic solution of $\mathcal{L}_\epsilon$ with $\sigma = \alpha + i\beta$ and $\beta \neq 0$, then for every $t \in \mathbb{R}$, $|\phi(t)| \neq |\psi(t)|$*

Proof. By contradiction, suppose that there is $t_0 \in \mathbb{R}$ such that $|\phi(t_0)| = |\psi(t_0)|$. Let $x_0 \in \mathbb{R}$ such that $\overline{\psi(t_0)} = -\mathrm{e}^{ix_0}\phi(t_0)$. Then the positive number $r_0 = \exp(x_0/2\beta)$ satisfies $r_0^{i\beta} = r_0^{-i\beta}\mathrm{e}^{ix_0}$ and consequently,

$$w(r_0, t_0) = r_0^\alpha(r_0^{i\beta}\phi(t_0) + \overline{r_0^{i\beta}\psi(t_0)}) = 0.$$

This contradicts Proposition 2.1. $\square$

This corollary implies that, for a given basic solution $w = r^\sigma\phi + \overline{r^\sigma\psi}$ with $\sigma \in \mathbb{C}\backslash\mathbb{R}$, one of the functions ϕ or ψ is dominant. That is, $|\phi(t)| > |\psi(t)|$ or $|\psi(t)| > |\phi(t)|$ for every $t \in \mathbb{R}$. Hence the winding number of w, $\mathrm{Ind}(w)$ is well defined and we have $\mathrm{Ind}(w) = \mathrm{Ind}(\phi)$ if $|\phi| > |\psi|$ and $\mathrm{Ind}(w) = \mathrm{Ind}(\overline{\psi})$ otherwise. When $\sigma \in \mathbb{R}$, we have $w = r^\sigma f(t)$ with f nowhere 0 and so $\mathrm{Ind}(w) = \mathrm{Ind}(f)$.

For a basic solution $w = r^\sigma\phi + \overline{r^\sigma\psi}$ with $|\phi| > |\psi|$, we will refer to σ as the exponent of w (or a spectral value of $\mathcal{L}_\epsilon$) and define the character of w by

$$\mathrm{Char}(w) = (\sigma, \mathrm{Ind}(w))$$

We will denote by $\mathrm{Spec}(\mathcal{L}_\epsilon)$ the set of exponents of basic solutions. That is

$$\mathrm{Spec}(\mathcal{L}_\epsilon) = \{\sigma \in \mathbb{C};\ \exists w,\ \mathrm{Char}(w) = (\sigma, \mathrm{Ind}(w))\} \tag{2.4}$$

Remark 2.1 When $\sigma \in \mathbb{C}\backslash\mathbb{R}$ and $w = r^\sigma\phi(t) + \overline{r^\sigma\psi(t)}$ is a basic solution with $\mathrm{Char}(w) = (\sigma, \mathrm{Ind}(\phi))$, the function $\widetilde{w} = r^\sigma(i\phi(t)) + \overline{r^\sigma i\psi(t)}$ is also a basic solution with $\mathrm{Char}(w) = \mathrm{Char}(\widetilde{w})$ and w, $\widetilde{w}$ are $\mathbb{R}$-independent.

When $\sigma = \tau \in \mathbb{R}$, and $w = r^\tau f(t)$ is a basic solution with $\mathrm{Char}(w) = (\tau, \mathrm{Ind}(f))$, it is not always the case that there is a second $\mathbb{R}$-independent basic solution with the same exponent τ. There is however a second $\mathbb{R}$-independent basic solution $\widetilde{w} = r^{\tau'}g(t)$ with the same winding number ($\mathrm{Ind}(f) = \mathrm{Ind}(g)$) but with a different exponent τ' (see Proposition 2.3).

The following proposition follows from the constancy of the winding number under continuous deformations.

Proposition 2.2 *Let $w_\epsilon(r,t) = r^{\sigma(\epsilon)}\phi(t,\epsilon) + \overline{r^{\sigma(\epsilon)}\psi(t,\epsilon)}$ be a continuous family of basic solutions of $\mathcal{L}_\epsilon$ with $\epsilon \in I$, where $I \subset \mathbb{R}$ is an interval. Then* $\mathrm{Char}(w_\epsilon)$ *depends continuously on ϵ and* $\mathrm{Ind}(w_\epsilon)$ *is constant.*

2.2 The spectral equation and $\mathrm{Spec}(\mathcal{L}_0)$

We use the 2×2 system of ordinary differential equations to obtain an equation for the spectral values in terms of the monodromy matrix. Results about the CR equation (1.9) are then used to list the properties of $\mathrm{Spec}(\mathcal{L}_0)$.

In order for a function $w(r,t) = r^\sigma\phi(t) + \overline{r^\sigma\psi(t)}$ to be a basic solution of $\mathcal{L}_\epsilon$, the 2π-periodic and $\mathbb{C}^2$-valued function $V(t) = \begin{pmatrix} \phi(t) \\ \psi(t) \end{pmatrix}$ must solve the periodic system of differential equations

$$\dot{V} = \mathbf{M}(t,\sigma,\epsilon)V \tag{$\mathrm{E}_{\sigma,\epsilon}$}$$

where

$$\mathbf{M}(t,\sigma,\epsilon) = \begin{pmatrix} i\dfrac{\sigma - \lambda_\epsilon\nu}{\lambda_\epsilon} & \dfrac{c(t)}{\lambda_\epsilon} \\ \dfrac{\overline{c(t)}}{\overline{\lambda_\epsilon}} & -i\dfrac{\sigma - \overline{\lambda_\epsilon}\nu}{\overline{\lambda_\epsilon}} \end{pmatrix}$$

Note that since $\mathbf{M}$ is linear in $\sigma \in \mathbb{C}$ and real analytic in $\epsilon \in \mathbb{R}$, then any solution $V(t,\sigma,\epsilon)$ is an entire function in σ and real analytic in ϵ. The fundamental matrix of $(\mathrm{E}_{\sigma,\epsilon})$ is the 2×2 matrix $\mathbf{V}(t,\sigma,\epsilon)$ satisfying

$$\dot{\mathbf{V}} = \mathbf{M}(t,\sigma,\epsilon)\mathbf{V}, \qquad \mathbf{V}(0,\sigma,\epsilon) = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix. We know from Floquet theory that

$$\mathbf{V}(t,\sigma,\epsilon) = \mathbf{P}(t,\sigma,\epsilon)\exp(t\mathbf{K}(\sigma,\epsilon))$$

where $\mathbf{P}$ is a 2π-periodic matrix (in t) and $\mathbf{P}$ and $\mathbf{K}$ are entire in σ and real analytic in ϵ. The monodromy matrix of $(\mathrm{E}_{\sigma,\epsilon})$ is

$$\mathbf{B}(\sigma,\epsilon) = \mathbf{V}(2\pi,\sigma,\epsilon) = \exp(2\pi\mathbf{K}(\sigma,\epsilon))\,.$$

The Liouville-Jacobi formula gives

$$\det(\mathbf{V}(t,\sigma,\epsilon)) = \exp\left(\int_0^t \mathrm{tr}(\mathbf{M}(s,\sigma,\epsilon))ds\right) = \exp\left(\frac{2b\epsilon}{|\lambda_\epsilon|^2}\sigma t\right),$$

where $\det(A)$ and $\mathrm{tr}(A)$ denote the determinant and the trace of the matrix A. Hence,

$$\det(\mathbf{B}(\sigma,\epsilon)) = \exp\left(\frac{4\pi b\epsilon}{|\lambda_\epsilon|^2}\sigma\right)\,. \tag{2.5}$$

In order for system $(\mathrm{E}_{\sigma,\epsilon})$ to have a periodic solution, the corresponding monodromy matrix $\mathbf{B}$ must have 1 as an eigenvalue. Thus σ must solve the spectral equation

$$1 - \mathrm{tr}(\mathbf{B}(\sigma,\epsilon)) + \det(\mathbf{B}(\sigma,\epsilon)) = 0.$$

or equivalently, $F(\sigma,\epsilon) = 0$, where

$$F(\sigma,\epsilon) = \mathrm{tr}(\mathbf{B}(\sigma,\epsilon)) - 1 - \exp\left(\frac{4\pi b\epsilon}{|\lambda_\epsilon|^2}\sigma\right)\,. \tag{2.6}$$

We first verify that $\mathrm{Spec}(\mathcal{L}_\epsilon)$ is a discrete set.

Lemma 2.1 *For every* $\epsilon \in \mathbb{R}$, $\operatorname{Spec}(\mathcal{L}_\epsilon)$ *is a discrete subset of* $\mathbb{C}$.

Proof. By contradiction, suppose that there exists $\epsilon_0 \in \mathbb{R}$ such that $\operatorname{Spec}(\mathcal{L}_{\epsilon_0})$ has an accumulation point in $\mathbb{C}$. This means that the roots of the solutions of the spectral equation $F(\sigma, \epsilon_0) = 0$ have an accumulation point. Since F is an entire function, then $F(\sigma, \epsilon_0) \equiv 0$. Thus, $\operatorname{Spec}(\mathcal{L}_{\epsilon_0}) = \mathbb{C}$. Let $\begin{pmatrix} \phi(t,\sigma) \\ \psi(t,\sigma) \end{pmatrix}$ be a continuous family of periodic solutions of $(\mathrm{E}_{\sigma,\epsilon_0})$. By Proposition 2.2, we can assume that $|\phi| > |\psi|$ for every $\sigma \in \mathbb{R}^+ + i\mathbb{R}$ and that $\operatorname{Ind}(\phi) = j_0$ (is constant). Now the first equation of $(\mathrm{E}_{\sigma,\epsilon_0})$ gives

$$\frac{\lambda_{\epsilon_0}}{2\pi i} \int_0^{2\pi} \frac{\dot{\phi}(t,\sigma)}{\phi(t,\sigma)} dt = \sigma - \lambda_{\epsilon_0}\nu + \frac{1}{2\pi i} \int_0^{2\pi} c(t) \frac{\psi(t,\sigma)}{\phi(t,\sigma)} dt \,.$$

That is,

$$\sigma = \lambda_{\epsilon_0}(j_0 + \nu) - \frac{1}{2\pi i} \int_0^{2\pi} c(t) \frac{\psi(t,\sigma)}{\phi(t,\sigma)} dt \qquad \forall \sigma \in \mathbb{R}^+ + i\mathbb{R}\,.$$

This is a contradiction since $\left| c\dfrac{\psi}{\phi} \right| < |c|$ $\square$

The following proposition describes the spectrum of $\mathcal{L}_0$.

Proposition 2.3 *For every* $j \in \mathbb{Z}$, *there exist* $\tau_j^\pm \in \mathbb{R}$ *with* $\tau_j^- \le \tau_j^+$ *and* $f_j^\pm \in C^{k+1}(\mathbb{S}^1, \mathbb{C})$, *such that* $w_j^\pm(r,t) = r^{\tau_j^\pm} f_j^\pm(t)$ *are* $\mathbb{R}$*-independent basic solution of* $\mathcal{L}_0$ *with*

$$\operatorname{Char}(w_j^\pm) = (\tau_j^\pm, j)\,.$$

Furthermore, $\operatorname{Spec}(\mathcal{L}_0) = \{\tau_j^\pm,\ j \in \mathbb{Z}\}$,

$$\cdots < \tau_{-1}^- \le \tau_{-1}^+ < \tau_0^- \le \tau_0^+ < \tau_1^- \le \tau_1^+ < \cdots$$

with $\lim\limits_{j\to-\infty} \tau_j^\pm = -\infty$, $\lim\limits_{j\to\infty} \tau_j^\pm = \infty$

Proof. We have here $\lambda_0 = a > 0$. The pushforward of the equation $\mathcal{L}_0 w = 0$ via the first integral $z = r^a \mathrm{e}^{it}$ of L_0 gives a CR equation with a singularity of the form studied in [9]. The spectral values $\tau_j^\pm$ of the CR equations are as in the proposition. It remains only to verify that $\mathcal{L}_0$ (or its equivalent CR equation) has no complex spectral values. The Laurent series representation for solutions of the CR equation (see [9]) imply that any solution of $\mathcal{L}_0 w = 0$ can be written as

$$w(r,t) = \sum_{j\in\mathbb{Z}} c_j^- r^{\tau_j^-} f_j^-(t) + c_j^+ r^{\tau_j^+} f_j^+(t)$$

with $c_j^\pm \in \mathbb{R}$. Now, if $w = r^\sigma \phi(t) + \overline{r^\sigma \psi(t)}$ is a basic solution of $\mathcal{L}_0$, then it follows at once, from the series representation, that σ is one of the $\tau_j^\pm$'s $\square$

2.3 Existence of basic solutions

We use the spectral equation together with Proposition 2.3 to show the existence of basic solutions for $\mathcal{L}_\epsilon$ with any given winding number. More precisely, we have the following proposition.

Proposition 2.4 *For every $j \in \mathbb{Z}$, there exists $\sigma_j^\pm(\epsilon) \in \mathrm{Spec}(\mathcal{L}_\epsilon)$ such that $\sigma_j^\pm(\epsilon)$ depends continuously on $\epsilon \in \mathbb{R}$, $\sigma_j^\pm(0) = \tau_j^\pm$, and the corresponding basic solution*

$$w_j^\pm(r,t,\epsilon) = r^{\sigma_j^\pm(\epsilon)}\phi(t,\epsilon) + \overline{r^{\sigma_j^\pm(\epsilon)}\psi(t,\epsilon)}$$

is continuous in ϵ and $\mathrm{Char}(w_j^\pm) = (\sigma_j^\pm(\epsilon), j)$.

Proof. For a given $j \in \mathbb{Z}$, it follows from Proposition 2.3 that the monodromy matrix $\mathbf{B}(\tau_j^\pm, 0)$ admits 1 as an eigenvalue. Since the spectral function $F(\sigma,\epsilon)$ given by (2.6) is entire in $\mathbb{C}\times\mathbb{R}$ and since $F(\tau_j^\pm, 0) = 0$, then $F(\sigma,\epsilon) = 0$ defines an analytic variety $\mathcal{V}$ in $\mathbb{C}\times\mathbb{R}$ passing through the points $(\tau_j^\pm, 0)$. The variable ϵ can be taken as a parameter for a branch of $\mathcal{V}$ through the point $(\tau_j^\pm, 0)$. This means that the equation $F(\sigma,\epsilon) = 0$ has a solution $\sigma = g(\epsilon) \in \mathbb{C}$, with g continuous and $g(0) = \tau_j^\pm$. In fact, g is real analytic except at isolated points. The matrix $\mathbf{B}(g(\epsilon),\epsilon)$ is continuous and has 1 as an eigenvalue for every ϵ. Let $E_0^\pm$ be an eigenvector of $\mathbf{B}(\tau_j^\pm, 0)$ with eigenvalue 1. We can select a continuous vector $E^\pm(\epsilon) \in \mathbb{C}^2$ such that

$$\mathbf{B}(g(\epsilon),\epsilon)E^\pm(\epsilon) = E^\pm(\epsilon) \quad \text{and} \quad E^\pm(0) = E_0^\pm$$

Let $V(t,\epsilon) = \mathbf{V}(t, g(\epsilon), \epsilon)E^\pm(\epsilon)$. Then $V(t,\epsilon)$ is a periodic solution of the equation $(\mathrm{E}_{g(\epsilon),\epsilon})$. If we set $V(t,\epsilon) = \begin{pmatrix} \phi_j^\pm(t,\epsilon) \\ \psi_j^\pm(t,\epsilon) \end{pmatrix}$, then

$$w(r,t,\epsilon) = r^{g(\epsilon)}\phi_j^\pm(t,\epsilon) + \overline{r^{g(\epsilon)}\psi_j^\pm(t,\epsilon)}$$

is a basic solution of $\mathcal{L}_\epsilon$ and it depends continuously on ϵ. Since for $\epsilon = 0$, $w(r,t,0)$ has character $(\tau_j^\pm, j)$, then by Proposition 2.2, the character of $w(r,t,\epsilon)$ is either $(g(\epsilon), j)$ if $|\phi_j^\pm| > |\psi_j^\pm|$ or $(\overline{g(\epsilon)}, j)$ if $|\phi_j^\pm| < |\psi_j^\pm|$. In the first case, $\sigma_j^\pm(\epsilon) = g(\epsilon) \in \mathrm{Spec}(\mathcal{L}_\epsilon)$ and, in the second, $\sigma_j^\pm(\epsilon) = \overline{g(\epsilon)} \in \mathrm{Spec}(\mathcal{L}_\epsilon)$
□

2.4 Properties of the fundamental matrix of $(E_{\sigma,\epsilon})$

We prove some symmetry properties of the fundamental matrix and of the monodromy matrix that will be used shortly.

Proposition 2.5 *There exist functions $f(t,\sigma,s)$, $g(t,\sigma,s)$ of class C^{k+1} in $t\in\mathbb{R}$, analytic in $(\sigma,s)\in\mathbb{C}\times\mathbb{R}$, such that the fundamental matrix $\mathbf{V}(t,\sigma,\epsilon)$ of $(\mathrm{E}_{\sigma,\epsilon})$ has the form*

$$\mathbf{V}(t,\sigma,\epsilon)=\begin{pmatrix} f(t,\sigma,\epsilon^2) & \overline{\lambda_\epsilon g(t,\overline{\sigma},\epsilon^2)} \\ \lambda_\epsilon g(t,\sigma,\epsilon^2) & \overline{f(t,\overline{\sigma},\epsilon^2)} \end{pmatrix}\exp\left(\frac{\epsilon bt}{|\lambda_\epsilon|^2}\sigma\right). \tag{2.7}$$

Furthermore, f and g satisfy

$$f(t,\sigma,\epsilon^2)\overline{f(t,\overline{\sigma},\epsilon^2)}-|\lambda_\epsilon|^2 g(t,\sigma,\epsilon^2)\overline{g(t,\overline{\sigma},\epsilon^2)}\equiv 1 \tag{2.8}$$

Proof. If we use the substitution $V=Z\exp\left(\dfrac{\epsilon b\sigma t}{|\lambda_\epsilon|^2}\right)$ in equation $(\mathrm{E}_{\sigma,\epsilon})$, then the system for Z is

$$\dot{Z}=\mathbf{A}(t,\sigma,\epsilon^2)Z \tag{2.9}$$

with

$$\mathbf{A}(t,\sigma,\epsilon^2)=\begin{pmatrix} i\mu & \overline{\lambda_\epsilon d(t,\epsilon^2)} \\ \lambda_\epsilon d(t,\epsilon^2) & -i\mu \end{pmatrix}$$

and where

$$\mu=\frac{a\sigma}{a^2+b^2\epsilon^2}-\nu\,,\quad \text{and}\quad d(t,\epsilon^2)=\frac{\overline{c(t)}}{a^2+b^2\epsilon^2}$$

The fundamental matrix $\mathbf{Z}(t,\sigma,\epsilon^2)$ of (2.9) with $\mathbf{Z}(0,t,\epsilon^2)=\mathbf{I}$ is therefore of class C^{k+1} in t and analytic in (σ,s) with $s=\epsilon^2$.

For functions $F(t,\mu,\epsilon^2)$ and $G(t,\mu,\epsilon^2)$, with $\mu\in\mathbb{R}$, we use the notation

$$\mathbf{D}_F=\begin{pmatrix} F & 0 \\ 0 & \overline{F} \end{pmatrix}\quad\text{and}\quad \mathbf{J}_{\lambda_\epsilon G}=\begin{pmatrix} 0 & \overline{\lambda_\epsilon G} \\ \lambda_\epsilon G & 0 \end{pmatrix}$$

With this notation, system (2.9) has the form

$$\dot{Z}=\left(\mathbf{D}_{i\mu}+\mathbf{J}_{\lambda_\epsilon d}\right)Z \tag{2.10}$$

Note that we have the following relations

$$\begin{array}{ll} \mathbf{D}_{F_1}\mathbf{D}_{F_2}=\mathbf{D}_{F_1F_2}, & \mathbf{D}_F\mathbf{J}_{\lambda_\epsilon G}=\mathbf{J}_{\lambda_\epsilon\overline{F}G}, \\ \mathbf{J}_{\lambda_\epsilon G}\mathbf{D}_F=\mathbf{J}_{\lambda_\epsilon FG}, & \mathbf{J}_{\lambda_\epsilon G_1}\mathbf{J}_{\lambda_\epsilon G_2}=\mathbf{D}_{|\lambda_\epsilon|^2\overline{G_1}G_2} \end{array}$$

The fundamental matrix $\mathbf{Z}$ is obtained as the limit, $\mathbf{Z}=\lim_{k\to\infty}\mathbf{Z}_k$, where the matrices $\mathbf{Z}_k(t,\sigma,\epsilon)$ are defined inductively by $\mathbf{Z}_0=\mathbf{I}$ and

$$\mathbf{Z}_{k+1}(t,\sigma,\epsilon)=\mathbf{I}+\int_0^t\left(\mathbf{D}_{i\mu}+\mathbf{J}_{\lambda_\epsilon d}\right)\mathbf{Z}_k(s,\sigma,\epsilon)ds$$

Now, we prove by induction that $\mathbf{Z}_k = \mathbf{D}_{F_k} + \mathbf{J}_{\lambda_\epsilon G_k}$, where F_k and G_k are polynomials in the variable μ, analytic and even in the variable ϵ. The claim is obviously true for $\mathbf{Z}_0 = \mathbf{I}$. Suppose that $\mathbf{Z}_k$ has the desired property for $k = 0, \cdots, n$, then

$$\begin{aligned} \mathbf{Z}_{n+1}(t,\sigma,\epsilon) &= \mathbf{I} + \int_0^t (\mathbf{D}_{i\mu} + \mathbf{J}_{\lambda_\epsilon d})\,(\mathbf{D}_{F_n} + \mathbf{J}_{\lambda_\epsilon G_n})\,ds \\ &= \mathbf{D}_{F_{n+1}} + \mathbf{J}_{\lambda_\epsilon G_{n+1}} \end{aligned}$$

where

$$\begin{aligned} F_{n+1}(t,\mu,\epsilon^2) &= 1 + \int_0^t \left(i\mu F_n(s,\mu,\epsilon^2) + c(s)G_n(s,\mu,\epsilon^2)\right) ds \\ G_{n+1}(t,\mu,\epsilon^2) &= -\int_0^t \left(i\mu G_n(s,\mu,\epsilon^2) + \overline{c(s)}F_n(s,\mu,\epsilon^2)\right) ds \end{aligned}$$

By taking the limit as $k \to \infty$, we get the fundamental matrix

$$\mathbf{Z}(t,\mu,\epsilon) = \mathbf{D}_{F_0(t,\mu,\epsilon^2)} + \mathbf{J}_{\lambda_\epsilon G_0(t,\mu,\epsilon^2)} \tag{2.11}$$

where F_0 and G_0 are entire functions with respect to the real parameters μ and ϵ^2. The fundamental matrix $\mathbf{Z}(t,\sigma,\epsilon)$ of (2.9) is therefore $\mathbf{D}_{f(t,\sigma,\epsilon^2)} + \mathbf{J}_{\lambda_\epsilon g(t,\sigma,\epsilon^2)}$, where f and g are the holomorphic extensions of F_0 and G_0 (obtained by replacing μ by $\dfrac{a\sigma}{a^2+b^2\epsilon^2} - \nu$, with $\sigma \in \mathbb{C}$). The proposition follows immediately, since $\mathbf{V} = \mathbf{Z}\exp\left(\dfrac{\epsilon b\sigma t}{|\lambda_\epsilon|^2}\right)$ $\square$

A direct consequence of expression (2.7) of the proposition is the following

Proposition 2.6 *Let $\mathbf{V}(t,\sigma,\epsilon)$ be the fundamental matrix of* $(\mathrm{E}_{\sigma,\epsilon})$*, then the fundamental matrix of equation* $(\mathrm{E}_{\overline{\sigma},\epsilon})$ *is*

$$\mathbf{V}(t,\overline{\sigma},\epsilon) = \mathbf{J}\overline{\mathbf{V}(t,\sigma,\epsilon)}\mathbf{J} \tag{2.12}$$

where $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The monodromy matrix of $(\mathrm{E}_{\sigma,\epsilon})$ has the form

$$\mathbf{B}(\sigma,\epsilon) = \begin{pmatrix} p(\sigma,\epsilon^2) & \overline{\lambda_\epsilon q(\overline{\sigma},\epsilon^2)} \\ \lambda_\epsilon q(\sigma,\epsilon^2) & \overline{p(\overline{\sigma},\epsilon^2)} \end{pmatrix} \exp\left(\frac{2\pi\epsilon b}{a^2+b^2\epsilon^2}\sigma\right) \tag{2.13}$$

with $p(\sigma,\epsilon^2) = f(2\pi,\sigma,\epsilon^2)$, $q(\sigma,\epsilon^2) = g(2\pi,\sigma,\epsilon^2)$ satisfying

$$p(\sigma,\epsilon^2)\overline{p(\overline{\sigma},\epsilon^2)} - |\lambda_\epsilon|^2 q(\sigma,\epsilon^2)\overline{q(\overline{\sigma},\epsilon^2)} \equiv 1 \tag{2.14}$$

Note that

$$\det(\mathbf{B}(\sigma,\epsilon)) = \exp\left(\frac{4\pi\epsilon b}{a^2+b^2\epsilon^2}\sigma\right) \tag{2.15}$$

We denote by $\mathrm{Spec}(\mathbf{B}(.,\epsilon))$ the set of spectral values of $\mathbf{B}(.,\epsilon)$, i.e.

$$\mathrm{Spec}(\mathbf{B}(.,\epsilon)) = \{\sigma \in \mathbb{C};\ \det(\mathbf{B}(\sigma,\epsilon)-\mathbf{I}) = 0\}.$$

It follows from (2.13) and Proposition 2.6 that

$$\mathrm{Spec}(\mathbf{B}(.,\epsilon)) = \mathrm{Spec}(\mathbf{B}(.,-\epsilon)) = \overline{\mathrm{Spec}(\mathbf{B}(.,\epsilon))} \tag{2.16}$$

An element $\sigma \in \mathrm{Spec}(\mathbf{B}(.,\epsilon))$ is said to be a simple (or a double) spectral value if the corresponding eigenspace has dimension 1 (or 2).

The spectral function F defined in (2.6) takes form

$$F(\sigma,\epsilon^2) = p(\sigma,\epsilon^2) + \overline{p(\overline{\sigma},\epsilon^2)} - 2\cosh\left(\frac{2\pi\epsilon b}{a^2+b^2\epsilon^2}\sigma\right) \tag{2.17}$$

Thus if $F(\sigma,\epsilon^2)=0$, then $F(\overline{\sigma},\epsilon)=0$. We have therefore

Corollary 2.0.2 *If $\sigma \in \mathrm{Spec}(\mathcal{L}_\epsilon)$, then σ or $\overline{\sigma} \in \mathrm{Spec}(\mathcal{L}_{-\epsilon})$.*

2.5 The system of equations for the adjoint operator $\mathcal{L}_\epsilon^*$

The properties of the fundamental matrix of system $\mathrm{E}_{\sigma,\epsilon}$ will be used to obtain those for the adjoint operator. The system of ordinary differential equations for the adjoint operator $\mathcal{L}_\epsilon^*$ given in (1.7) is

$$\dot{V} = \widetilde{\mathbf{M}}(t,\mu,\epsilon)V \tag{$\widetilde{\mathrm{E}_{\mu,\epsilon}}$}$$

where

$$\widetilde{\mathbf{M}}(t,\mu,\epsilon) = \begin{pmatrix} i\dfrac{\mu+\lambda_\epsilon\nu}{\lambda_\epsilon} & -\dfrac{\overline{c(t)}}{\lambda_\epsilon} \\ -\dfrac{c(t)}{\overline{\lambda_\epsilon}} & -i\dfrac{\mu+\overline{\lambda_\epsilon}\nu}{\overline{\lambda_\epsilon}} \end{pmatrix} \tag{2.18}$$

Thus, if $V(t) = \begin{pmatrix} X(t) \\ Z(t) \end{pmatrix}$ is a periodic solution of $(\widetilde{\mathrm{E}_{\mu,\epsilon}})$, then $w(r,t) = r^\mu X(t) + \overline{r^\mu Z(t)}$ is a basic solution of $\mathcal{L}_\epsilon^*$.

The relation between the fundamental matrices of this system and those for $\mathrm{E}_{\sigma,\epsilon}$ is given by the following proposition.

Proposition 2.7 *The fundamental matrix of $(\widetilde{\mathrm{E}_{\mu,\epsilon}})$ is*

$$\widetilde{\mathbf{V}}(t,\mu,\epsilon) = \mathbf{D}\overline{\mathbf{V}(t,-\overline{\mu},-\epsilon)}\mathbf{D} \tag{2.19}$$

where $\mathbf{V}(t,-\overline{\mu},-\epsilon)$ is the fundamental matrix of $(\mathrm{E}_{-\overline{\mu},-\epsilon})$ and $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Proof. If $V = \begin{pmatrix} X \\ Z \end{pmatrix}$ solves $(\widetilde{\mathrm{E}_{\mu,\epsilon}})$, then $\overline{\mathbf{D}V} = \begin{pmatrix} \overline{X} \\ -\overline{Z} \end{pmatrix}$ solves the equation $(\mathrm{E}_{-\overline{\mu},-\epsilon})$. Therefore, if $\begin{pmatrix} X_1 & X_2 \\ Z_1 & Z_2 \end{pmatrix}$ is a fundamental matrix of $(\widetilde{\mathrm{E}_{\mu,\epsilon}})$, then $\begin{pmatrix} \overline{X_1} & -\overline{X_2} \\ -\overline{Z_1} & \overline{Z_2} \end{pmatrix}$ is a fundamental matrix of $(\mathrm{E}_{-\overline{\mu},-\epsilon})$ □

Immediate consequences are the following corollaries.

Corollary 2.0.3 *The monodromy matrix of* $(\widetilde{\mathrm{E}_{\mu,\epsilon}})$ *is*

$$\widetilde{\mathbf{B}}(\mu,\epsilon) = \mathbf{D}\overline{\mathbf{B}(-\overline{\mu},-\epsilon)}\mathbf{D} \tag{2.20}$$

where $\mathbf{B}(\sigma,\epsilon)$ *is the monodromy matrix of* $(\mathrm{E}_{\sigma,\epsilon})$*. Furthermore, if* $\sigma \in \mathrm{Spec}(\mathbf{B}(.,\epsilon))$ *and* $\mathbf{B}(\sigma,\epsilon)E = E$*, then* $-\overline{\sigma} \in \mathrm{Spec}(\widetilde{\mathbf{B}}(.,-\epsilon))$ *and* $\widetilde{\mathbf{B}}(-\overline{\sigma},-\epsilon)\mathbf{D}\overline{E} = \mathbf{D}\overline{E}$*.*

Corollary 2.0.4 *If* $\sigma \in \mathrm{Spec}(\mathcal{L}_\epsilon)$*, then either* $-\sigma \in \mathrm{Spec}(\mathcal{L}^*_\epsilon)$ *or* $-\overline{\sigma} \in \mathrm{Spec}(\mathcal{L}^*_\epsilon)$

2.6 Continuation of a simple spectral value

We start from a simple spectral value, when $\epsilon = 0$, and use the properties of the fundamental matrix to obtain the behavior of $\sigma(\epsilon)$ for ϵ near 0.

Proposition 2.8 *Suppose that* $\tau \in \mathrm{Spec}(\mathbf{B}(.,0))$ *and that* τ *is simple. Then there exist* $\delta > 0$ *and a unique function* $\sigma \in C^0([-\delta,\ \delta],\mathbb{R})$ *such that* $\sigma(0) = \tau$ *and* $\sigma(\epsilon) \in \mathrm{Spec}(\mathbf{B}(.,\epsilon))$ *for every* $\epsilon \in [-\delta,\ \delta]$

Proof. The matrix $\mathbf{B}(\tau,0)$ has a single eigenvector U (up to a multiple) with eigenvalue 1. Since $\det(\mathbf{B}(\tau,0)) = 1$ (see (2.13) and (2.14)), then $\mathbf{B}(\tau,0)$ is similar to the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let $\mathbf{V}(t,\sigma,\epsilon)$ be the fundamental matrix of $(\mathrm{E}_{\sigma,\epsilon})$. The function

$$\begin{pmatrix} \phi(t) \\ \psi(t) \end{pmatrix} = \mathbf{V}(t,\tau,0)U,$$

generates all periodic solutions of $(\mathrm{E}_{\tau,0})$. First, we show that we can find a generator of the form $\begin{pmatrix} f(t) \\ \overline{f}(t) \end{pmatrix}$ for some function f. For this, note that since $\lambda_0 = a \in \mathbb{R}$, then it follows from (2.2) that $\begin{pmatrix} \overline{\psi}(t) \\ \overline{\phi}(t) \end{pmatrix}$ is also a periodic solution of $(\mathrm{E}_{\tau,0})$. Hence, there exists $c \in \mathbb{C}$, $|c| = 1$ such that $\overline{\psi}(t) = c\phi(t)$ and $\overline{\phi}(t) = c\psi(t)$. If $c = 1$, then we can take $f = \phi$, if $c \neq -1$, we can take

$f = \phi + \overline{\psi}$, and if $c = -1$, we take $f = i\phi$. The vector $U_0 = \begin{pmatrix} f(0) \\ \overline{f}(0) \end{pmatrix}$ is the eigenvector of $\mathbf{B}(\tau, 0)$ that generates the solution $\begin{pmatrix} f(t) \\ \overline{f}(t) \end{pmatrix}$.

We know from Proposition 2.4 that the spectral function $F(\sigma, \epsilon^2)$ given in (2.6) has a root $\sigma(\epsilon)$ with $\sigma(0) = \tau$. Furthermore, $\sigma(\epsilon)$ is real analytic in a neighborhood of $0 \in \mathbb{R}$, except possibly at $\epsilon = 0$. Now we show that there is only one such function in a neighborhood of 0 and that it is real-valued. Starting from U_0, we can find a continuous vector $U(\epsilon) \in \mathbb{C}^2$ with $U(0) = U_0$ and such that $\mathbf{B}(\sigma(\epsilon), \epsilon)U(\epsilon) = U(\epsilon)$. The function

$$\begin{pmatrix} \phi(t, \epsilon) \\ \psi(t, \epsilon) \end{pmatrix} = \mathbf{V}(t, \sigma(\epsilon), \epsilon)U(\epsilon)$$

is a periodic solution of $(\mathrm{E}_{\sigma(\epsilon).\epsilon})$ such that $\phi(t, 0) = f(t)$ and $\psi(t, 0) = \overline{f}(t)$. It follows from Corollary 2.0.4 that $-\tau \in \mathrm{Spec}(\widetilde{\mathbf{B}}(., 0))$ and $-\overline{\sigma(\epsilon)} \in \mathrm{Spec}(\widetilde{\mathbf{B}}(., \epsilon))$. Note that if $V(t)$ is a periodic solution of $(\mathrm{E}_{\tau,0})$, then $\mathbf{D}\overline{V}(t)$ is a periodic solution of $(\widetilde{\mathrm{E}}_{-\tau,0})$. Thus, $\begin{pmatrix} \overline{f}(t) \\ -f(t) \end{pmatrix}$ solves $(\widetilde{\mathrm{E}}_{-\tau,0})$. Let $U_1(\epsilon) \in \mathbb{C}^2$ be a continuous eigenvector of $\mathbf{B}(\sigma(\epsilon), -\epsilon)$ such that $U_1(0) = U_0$. Set $U^\star(\epsilon) = \mathbf{D}\overline{U_1(\epsilon)}$. Then, it follows from Corollary 2.0.3, that

$$\widetilde{\mathbf{B}}(-\overline{\sigma(\epsilon)}, \epsilon)U^\star(\epsilon) = \mathbf{D}\overline{\mathbf{B}(\sigma(\epsilon), -\epsilon)}\mathbf{D}\mathbf{D}\overline{U_1(\epsilon)} = \mathbf{D}\overline{U_1(\epsilon)} = U^\star(\epsilon).$$

Therefore,

$$\begin{pmatrix} X(t, \epsilon) \\ Z(t, \epsilon) \end{pmatrix} = \widetilde{\mathbf{V}}(t, -\overline{\sigma(\epsilon)}, \epsilon)U^\star(\epsilon)$$

is a periodic solution of $(\widetilde{\mathrm{E}}_{-\overline{\sigma(\epsilon)}.\epsilon})$ with $X(t, 0) = \overline{f}(t)$ and $Z(t, 0) = -f(t)$.

The corresponding basic solutions of $\mathcal{L}_\epsilon$ and $\mathcal{L}^*_\epsilon$ are respectively,

$$\begin{aligned} w_\epsilon(r, t) &= r^{\sigma(\epsilon)}\phi(t, \epsilon) + \overline{r^{\sigma(\epsilon)}\psi(t, \epsilon)}, \quad \text{and} \\ w^\star_\epsilon(r, t) &= r^{-\overline{\sigma(\epsilon)}}X(t, \epsilon) + r^{-\sigma(\epsilon)}\overline{Z(t, \epsilon)} \end{aligned}$$

We apply Green's formula (1.8) to the pair w_ϵ, $w^\star_\epsilon$ in the cylinder $A = [R_1,\ R_2] \times \mathbb{S}^1$ (with $0 < R_1 < R_2$) to get

$$\mathrm{Re}\left[\int_{\partial A} w_\epsilon(r, t)w^\star_\epsilon(r, t)\frac{dz_\epsilon}{z_\epsilon}\right] = 0\,.$$

That is,

$$\mathrm{Re}\left[\int_0^{2\pi} \left((R_2^{\sigma-\overline{\sigma}} - R_1^{\sigma-\overline{\sigma}})\phi X + (R_2^{\overline{\sigma}-\sigma} - R_1^{\overline{\sigma}-\sigma})\overline{\psi Z}\right) idt\right] = 0\,. \tag{2.21}$$

Suppose that $\sigma(\epsilon)$ is not $\mathbb{R}$-valued in a neighborhood of 0. Then $\sigma(\epsilon) = \alpha(\epsilon) + i\beta(\epsilon)$ with $\beta(\epsilon) > 0$ (or < 0) in a an interval $(0,\ \epsilon_0)$. If we set $p = \log R_2$ and $q = \log R_1$, we get

$$R_2^{\sigma-\overline{\sigma}} - R_1^{\sigma-\overline{\sigma}} = \mathrm{e}^{2i\beta p} - \mathrm{e}^{2i\beta q} = 2i\sin(\beta(p-q))\mathrm{e}^{i\beta(p+q)}$$

and (2.21) becomes (with $x = p+q$ arbitrary)

$$\mathrm{Re}\left[\int_0^{2\pi}\left(\mathrm{e}^{i\beta x}\phi(t,\epsilon)X(t,\epsilon) - i\mathrm{e}^{-i\beta x}\overline{\psi(t,\epsilon)Z(t,\epsilon)}\right)dt\right] = 0\,, \qquad (2.22)$$

Let $P(\epsilon) + iQ(\epsilon) = \displaystyle\int_0^{2\pi}\phi(t,\epsilon)X(t,\epsilon)dt$ and $R(\epsilon) + iS(\epsilon) = \displaystyle\int_0^{2\pi}\overline{\psi(t,\epsilon)Z(t,\epsilon)}dt$
From (2.22), we have

$$\cos(\beta(\epsilon)x)(P(\epsilon) - R(\epsilon)) - \sin(\beta(\epsilon)x)(Q(\epsilon) + S(\epsilon)) = 0, \qquad \forall x \in \mathbb{R}\,.$$

Therefore,

$$P(\epsilon) - R(\epsilon) = 0, \quad Q(\epsilon) + S(\epsilon) = 0, \qquad \forall \epsilon \in (0,\ \epsilon_0).$$

By continuity, we get $P(0) = R(0)$ and $Q(0) = -S(0)$. But,

$$\begin{aligned} P(0) + iQ(0) &= \int_0^{2\pi}\phi(t,0)X(t,0)dt = \int_0^{2\pi}|f(t)|^2dt \\ R(0) + iS(0) &= \int_0^{2\pi}\overline{\psi(t,0)Z(t,0)}dt = -\int_0^{2\pi}|f(t)|^2dt \end{aligned}$$

and it follows from $P(0) = R(0)$ that $\displaystyle\int_0^{2\pi}|f|^2dt = 0$. This is a contradiction since $f \neq 0$. This means that $\sigma(\epsilon)$ is an $\mathbb{R}$-valued function in a neighborhood of $\epsilon = 0$.

Now we show that $\sigma(\epsilon)$ is unique near $\epsilon = 0$. By contradiction, suppose that there is another real valued solution $\sigma_1(\epsilon)$, with $\sigma(\epsilon) < \sigma_1(\epsilon)$ in an interval $(0,\ \epsilon_0)$, and $\sigma(0) = \sigma_1(0) = \tau$. Let $\phi(t,\epsilon)$ and $\psi(t,\epsilon)$ be as above. Let $U'(\epsilon)$ be an eigenvector (with eigenvalue 1) of $\mathbf{B}(\sigma_1(\epsilon), -\epsilon)$ such that $U'(0) = -iU_0$, where U_0 is the eigenvector used above. Let $U_1^\star(\epsilon) = \mathbf{D}\overline{U'(\epsilon)}$. Then

$$\widetilde{\mathbf{B}}(-\sigma_1(\epsilon),\epsilon)U^\star(\epsilon) = U_1^\star(\epsilon) \quad \text{and} \quad U_1^\star(0) = i\mathbf{D}U_0\,.$$

To $U^\star(\epsilon)$ corresponds the 2π-periodic solution

$$\begin{pmatrix} X_1(t,\epsilon) \\ Z_1(t,\epsilon) \end{pmatrix} = \widetilde{\mathbf{V}}(t, -\overline{\sigma_1(\epsilon)}, \epsilon)U_1^\star(\epsilon)$$

of $(\widetilde{\mathrm{E}}_{-\overline{\sigma_1(\epsilon)}.\epsilon})$ with $X_1(t,0) = i\overline{f}(t)$ and $Z_1(t,0) = -if(t)$. The corresponding basic solution of $\mathcal{L}_\epsilon^*$ is

$$w_{1,\epsilon}^\star(r,t) = r^{-\sigma_1(\epsilon)}(X_1(t,\epsilon) + \overline{Z_1(t,\epsilon)})$$

The Green's formula, applied to the pair w_ϵ, $w^\star_{1,\epsilon}$ in the cylinder $(R_1,\ R_2)\times\mathbb{S}^1$, gives

$$\mathrm{Re}\left[\int_0^{2\pi}(R_2^{\sigma-\sigma_1}-R_1^{\sigma-\sigma_1})(\phi+\overline{\psi})(X_1+\overline{Z_1})idt\right]=0\,.$$

Thus,

$$\mathrm{Re}\left[\int_0^{2\pi}(\phi(t,\epsilon)+\overline{\psi(t,\epsilon)})(X_1(t,\epsilon)+\overline{Z_1(t,\epsilon)})idt\right]=0\qquad\forall\epsilon\in(0,\ \epsilon_0)\,.$$

By letting $\epsilon\to 0$, we get again $\displaystyle\int_0^{2\pi}|f(t)|^2dt=0$, which is a contradiction. This shows that $\sigma(\epsilon)$ is unique for ϵ near 0 $\square$

2.7 Continuation of a double spectral value

This time we study the behavior of $\sigma(\epsilon)$ when $\sigma(0)$ has multiplicity 2. Hence assume that $\tau\in\mathrm{Spec}(\mathbf{B}(.,0))$ has multiplicity 2. Therefore, $\mathbf{B}(\tau,0)=\mathbf{I}$. We start with the following proposition.

Proposition 2.9 *If* $\mathbf{B}(\tau,0)=\mathbf{I}$, *then* $\dfrac{\partial\mathrm{tr}\mathbf{B}}{\partial\sigma}(\tau,0)=0$ *and* $\dfrac{\partial^2\mathrm{tr}\mathbf{B}}{\partial\sigma^2}(\tau,0)\neq 0$.

The proof of this proposition makes use of the the following lemma.

Lemma 2.2 *Given* $M>0$, *there is a positive constant* C *such that*

$$(1+2x)(1+2y)-4\sqrt{xy(1+x)(1+y)}\geq C,\qquad\forall x,y\in[0,\ M]$$

Proof. Consider the function $g(x,y)=(1+2x)^2(1+2y)^2-16xy(1+x)(1+y)$. It can be easily verified that $g(x,y)\geq 1$ in the square $[0,\ M]^2$. This implies in turn that

$$(1+2x)(1+2y)-4\sqrt{xy(1+x)(1+y)}\geq\frac{1}{1+8M+8M^2}\quad\square$$

Proof of Proposition 2.9. Let $\mathbf{V}(t,\sigma,\epsilon)$ be the fundamental matrix of equation $(\mathrm{E}_{\sigma,\epsilon})$ given by (2.7). Its derivative $\mathbf{V}_\sigma$, with respect to σ, satisfies the system

$$\dot{\mathbf{V}}_\sigma=\mathbf{M}\mathbf{V}_\sigma+\mathbf{M}_\sigma\mathbf{V},\quad\mathbf{V}_\sigma(0,\sigma,\epsilon)=0\tag{2.23}$$

(the last condition follows from $\mathbf{V}(0,\sigma,\epsilon)=\mathbf{I}$). Note that

$$\mathbf{M}_\sigma=\mathbf{D}_{i/\lambda_\epsilon}=\begin{pmatrix}\dfrac{i}{\lambda_\epsilon} & 0\\ 0 & \dfrac{-i}{\overline{\lambda_\epsilon}}\end{pmatrix}.$$

We consider (2.23) as a nonhomogeneous system in $\mathbf{V}_\sigma$ and we get

$$\mathbf{V}_\sigma(t,\sigma,\epsilon) = \mathbf{V}(t,\sigma,\epsilon)\int_0^t \mathbf{V}(s,\sigma,\epsilon)^{-1}\mathbf{D}_{i/\lambda_\epsilon}\mathbf{V}(s,\sigma,\epsilon)ds\,. \tag{2.24}$$

By using formula (2.7), we have

$$\mathbf{V}^{-1}\mathbf{D}_{i/\lambda_\epsilon}\mathbf{V} = i\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22}\end{pmatrix}$$

where

$$\begin{aligned}
N_{11} &= \frac{f(t,\sigma,\epsilon^2)\overline{f(t,\overline{\sigma},\epsilon^2)}}{\lambda_\epsilon} + \lambda_\epsilon g(t,\sigma,\epsilon^2)\overline{g(t,\overline{\sigma},\epsilon^2)} \\
N_{12} &= \frac{2a}{\lambda_\epsilon}\overline{f(t,\overline{\sigma},\epsilon^2)g(t,\overline{\sigma},\epsilon^2)} \\
N_{21} &= -\frac{2a}{\overline{\lambda_\epsilon}}f(t,\sigma,\epsilon^2)g(t,\sigma,\epsilon^2) \\
N_{22} &= -\frac{f(t,\sigma,\epsilon^2)\overline{f(t,\overline{\sigma},\epsilon^2)}}{\overline{\lambda_\epsilon}} - \overline{\lambda_\epsilon}g(t,\sigma,\epsilon^2)\overline{g(t,\overline{\sigma},\epsilon^2)}
\end{aligned}$$

In particular

$$\mathrm{tr}(\mathbf{V}^{-1}\mathbf{D}_{i/\lambda_\epsilon}\mathbf{V})(t,\sigma,0) = iN_{11}(t,\sigma,0) + iN_{22}(t,\sigma,0) \equiv 0. \tag{2.25}$$

If we set $t = 2\pi$ in (2.24), we get

$$\frac{\partial\mathbf{B}}{\partial\sigma}(\sigma,\epsilon) = \mathbf{B}(\sigma,\epsilon)\int_0^{2\pi}\mathbf{V}(s,\sigma,\epsilon)^{-1}\mathbf{D}_{i/\lambda_\epsilon}\mathbf{V}(s,\sigma,\epsilon)ds \tag{2.26}$$

Since, $\mathbf{B}(\tau,0) = \mathbf{I}$, then it follows at once from (2.25) and (2.26) that $\dfrac{\partial\mathrm{tr}\mathbf{B}}{\partial\sigma}(\tau,0) = 0$. Now we compute $\mathbf{V}_{\sigma\sigma}$. We have

$$\dot{\mathbf{V}}_{\sigma\sigma} = \mathbf{M}\mathbf{V}_{\sigma\sigma} + 2\mathbf{D}_{i/\lambda_\epsilon}\mathbf{V}_\sigma\,, \quad \mathbf{V}_{\sigma\sigma}(0,\sigma,\epsilon) = 0 \tag{2.27}$$

and after integrating this nonhomogeneous system and using (2.24), we obtain

$$\begin{aligned}
\mathbf{V}_{\sigma\sigma}(t,\sigma,\epsilon) &= 2\mathbf{V}(t,\sigma,\epsilon)\int_0^t \mathbf{V}^{-1}\mathbf{D}_{i/\lambda_\epsilon}\mathbf{V}_\sigma(s,\sigma,\epsilon)ds \\
&= 2\mathbf{V}(t,\sigma,\epsilon)\int_0^t\int_0^s \mathbf{L}(s,\sigma,\epsilon)\mathbf{L}(u,\sigma,\epsilon)duds
\end{aligned} \tag{2.28}$$

where $\mathbf{L}(t,\sigma,\epsilon) = \mathbf{V}^{-1}\mathbf{D}_{i/\lambda_\epsilon}\mathbf{V}(t,\sigma,\epsilon)$. We have in particular

$$\mathbf{L}(t,\sigma,0) = \frac{i}{a}\begin{pmatrix} P(t,\sigma) & 2\overline{Q(t,\overline{\sigma})} \\ -2Q(t,\sigma) & -P(t,\sigma)\end{pmatrix} \tag{2.29}$$

where

$$\begin{aligned}
P(t,\sigma) &= f(t,\sigma,0)\overline{f(t,\overline{\sigma},0)} + a^2 g(t,\sigma,0)\overline{g(t,\overline{\sigma},0)} \\
Q(t,\sigma) &= af(t,\sigma,0)g(t,\sigma,0)
\end{aligned} \tag{2.30}$$

If we set $t = 2\pi$ in (2.28), we get

$$\mathbf{B}_{\sigma\sigma}(\sigma,\epsilon) = 2\mathbf{B}(\sigma,\epsilon)\int_0^{2\pi}\int_0^s \mathbf{L}(s,\sigma,\epsilon)\mathbf{L}(u,\sigma,\epsilon)duds$$

and since $\mathbf{B}(\tau, 0) = \mathbf{I}$, we have

$$\frac{\partial^2 \mathrm{tr}(\mathbf{B})}{\partial\sigma^2}(\tau,0) = 2\int_0^{2\pi}\int_0^s \mathrm{tr}(\mathbf{L}(s,\tau,0)\mathbf{L}(u,\tau,0))duds \tag{2.31}$$

It follows from (2.29) that

$$\frac{-a^2}{2}\mathrm{tr}(\mathbf{L}(s,\tau,0)\mathbf{L}(u,\tau,0)) = P(s,\tau)P(u,\tau) - 4\mathrm{Re}\left[Q(s,\tau)\overline{Q(u,\tau)}\right] \tag{2.32}$$

Let $g(t,\tau,0) = \dfrac{\rho(t)}{a}\mathrm{e}^{i\beta(t)}$ (thus, $\rho = a|g|$ and β is the argument of g) and then since f and g satisfy (2.8) we have $f(t,\tau,0) = (1+\rho(t)^2)^{1/2}\mathrm{e}^{i\alpha(t)}$. With this notation, the functions P and Q become

$$P(t,\tau) = 1 + 2\rho(t)^2 \quad \text{and} \quad Q(t,\tau) = \rho(t)\sqrt{1+\rho(t)^2}\mathrm{e}^{i(\alpha(t)+\beta(t))}$$

If we set $x = \rho(s)^2$, $y = \rho(u)^2$, and $\theta = \alpha(s) + \beta(s) - \alpha(u) - \beta(u)$, then formula (2.32) becomes

$$\frac{-a^2}{2}\mathrm{tr}(\mathbf{L}(s,\tau,0)\mathbf{L}(u,\tau,0)) = (1+2x)(1+2y) - 4\sqrt{xy(1+x)(1+y)}\cos\theta \tag{2.33}$$

Since, x, y are positive and bounded (g is bounded over the interval $[0,\ \pi]$), then Lemma 2.2 implies that there is a positive constant C such that

$$\mathrm{tr}(\mathbf{L}(s,\tau,0)\mathbf{L}(u,\tau,0)) \le -C \qquad \forall u, s \in [0,\ 2\pi].$$

Therefore, by (2.31), we have $\mathrm{tr}(\mathbf{B}_{\sigma\sigma}(\tau, 0) \neq 0$. This completes the proof of the Proposition $\square$

The behavior of the spectral values of $\mathcal{L}_\epsilon$ is given by the following proposition.

Proposition 2.10 *Suppose that* $\mathbf{B}(\tau, 0) = \mathbf{I}$. *Then there exists* $\epsilon_0 > 0$ *such that the spectral values of* $\mathcal{L}_\epsilon$ *through* τ *satisfy one of the followings:*

1. *there is a unique continuous function* $\sigma(\epsilon)$ *defined in* $[-\epsilon_0,\ \epsilon_0]$ *such that* $\sigma(\epsilon) \in \mathrm{Spec}(\mathcal{L}_\epsilon)$, $\sigma(\epsilon) \in \mathbb{C}\backslash\mathbb{R}$ *for* $\epsilon \neq 0$ *and* $\sigma(0) = \tau$*;*

2. *there are two continuous* $\mathbb{R}$*-valued functions* $\sigma_1(\epsilon)$, $\sigma_2(\epsilon)$ *defined in* $[-\epsilon_0,\ \epsilon_0]$, *such that* $\sigma_1(\epsilon), \sigma_2(\epsilon) \in \mathrm{Spec}(\mathcal{L}_\epsilon)$, $\sigma_1(\epsilon) < \sigma_2(\epsilon)$ *for* $\epsilon \neq 0$, *and* $\sigma_1(0) = \sigma_2(0) = \tau$*;*

3. *there is a unique continuous* $\mathbb{R}$*-valued function* $\sigma(\epsilon)$ *defined in* $[-\epsilon_0,\ \epsilon_0]$ *such that* $\sigma(\epsilon) \in \mathrm{Spec}(\mathcal{L}_\epsilon)$ *and* $\sigma(0) = \tau$

Proof. It follows from Proposition 2.9 that the spectral function $F(\sigma, \epsilon)$ defined in (2.6) satisfies

$$F(\tau, 0) = 0, \quad \frac{\partial F}{\partial \sigma}(\tau, 0) = 0, \quad \text{and} \quad \frac{\partial^2 F}{\partial \sigma^2}(\tau, 0) \neq 0\,.$$

Since F is analytic in both variables, then by the Weierstrass Preparation Theorem (see [6]) we can find analytic functions $G(\sigma, \epsilon)$, $A_0(\epsilon)$ and $A_1(\epsilon)$ with (σ, ϵ) near $(\tau, 0) \in \mathbb{C} \times \mathbb{R}$, such that $G(\tau, 0) \neq 0$, $A_1(0) = A_0(0) = 0$ and

$$F(\sigma, \epsilon) = G(\sigma, \epsilon)\left[(\sigma - \tau)^2 - 2A_1(\epsilon)(\sigma - \tau) + A_0(\epsilon)\right]\,.$$

Thus, there exists $\epsilon_0 > 0$ such that the roots of the spectral equation $F(\sigma, \epsilon) = 0$ in a neighborhood of $(\tau, 0)$ are given by the quadratic formula

$$\sigma_{1,2}(\epsilon) = \tau + A_1(\epsilon) \pm \sqrt{A_1^2(\epsilon) - A_0(\epsilon)}\,.$$

The conclusion of the proposition follows depending on the sign of the discriminant $A_1^2 - A_0$ □

2.8 Purely imaginary spectral value

We study here the behavior of the monodromy matrix at possible spectral value on the imaginary axis.

Proposition 2.11 *Suppose that for some* $\epsilon_0 \in \mathbb{R}^*$*, the operator* $\mathcal{L}_{\epsilon_0}$ *has a spectral value* σ_0 *of the form*

$$\sigma_0 = i\frac{|\lambda_{\epsilon_0}|^2}{2b\epsilon_0}k, \quad \text{with} \quad k \in \mathbb{Z}^*\,. \tag{2.34}$$

Then the monodromy matrix $\mathbf{B}(\sigma_0, \epsilon_0)$ *is similar to* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Proof. If σ_0 given by (2.34) is a spectral value, then $\det \mathbf{B}(\sigma_0, \epsilon_0) = 1$, by (2.5). Hence 1 is an eigenvalue of $\mathbf{B}(\sigma_0, \epsilon_0)$ and, a priori, it could have multiplicity 2. In which case $\mathbf{B}(\sigma_0, \epsilon_0) = \mathbf{I}$. We are going to show that this case does not happen. By contradiction, suppose that $\mathbf{B}(\sigma_0, \epsilon_0) = \mathbf{I}$. First we prove that $\mathrm{tr}(\mathbf{B}_\sigma(\sigma_0, \epsilon_0)) \neq 0$. From formulas (2.26), (2.25), and (2.8) we have

$$\frac{\partial \mathrm{tr}(\mathbf{B})}{\partial \sigma}(\sigma_0, \epsilon_0) = \int_0^{2\pi} m(s)ds$$

where

$$\begin{aligned} m(s) &= \frac{\overline{\lambda_{\epsilon_0}} - \lambda_{\epsilon_0}}{|\lambda_{\epsilon_0}|^2}\left[f(s,\sigma_0,\epsilon_0)\overline{f(s,\overline{\sigma_0},\epsilon_0)} - |\lambda_{\epsilon_0}|^2 g(s,\sigma_0,\epsilon_0)\overline{g(s,\overline{\sigma_0},\epsilon_0)}\right] \\ &= \frac{-2ib\epsilon_0^2}{|\lambda_{\epsilon_0}|^2} \end{aligned}$$

This shows that $\mathrm{tr}(\mathbf{B}_\sigma(\sigma_0,\epsilon_0)) \neq 0$. Hence, it follows that the spectral function satisfies

$$\frac{\partial F}{\partial \sigma}(\sigma_0,\epsilon_0) = \frac{\partial \mathrm{tr}(\mathbf{B})}{\partial \sigma}(\sigma_0,\epsilon_0) \neq 0\,.$$

By the implicit function theorem, the germ of the analytic variety $F(\sigma,\epsilon) = 0$ through (σ_0,ϵ_0) is smooth and there is a unique analytic function $\sigma(\epsilon)$ defined near ϵ_0 with $\sigma(\epsilon_0) = \sigma_0$ such that $F(\sigma(\epsilon),\epsilon) \equiv 0$. It follows that $\mathbf{B}(.,\epsilon)$ has a unique spectral value through (σ_0,ϵ_0).

Let $U_1(\epsilon)$ be a continuous eigenvector (with eigenvalue 1) of $\mathbf{B}(\sigma(\epsilon),\epsilon)$ defined in an interval $(\epsilon_0 - \delta,\ \epsilon_0 + \delta)$ for some $\delta > 0$. We can assume that $U_1(\epsilon_0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ with $\alpha \neq 0$. Now, consider the equation

$$G(\sigma,\epsilon,z) = (\mathbf{B}(\sigma,\epsilon) - \mathbf{I})\begin{pmatrix} z \\ 1 \end{pmatrix} = 0$$

in a neighborhood of $(\sigma_0,\epsilon_0,0) \in \mathbb{C}\times\mathbb{R}\times\mathbb{C}$. This equation defines a germ of an analytic variety of real dimension 1 in $\mathbb{C}\times\mathbb{R}\times\mathbb{C}$ that passes through the point $(\sigma_0,\epsilon_0,0)$. Therefore, there exists $\delta_1 > 0$ and continuous functions $\sigma'(\epsilon)$ and $z(\epsilon)$ defined in $[\epsilon_0-\delta_1,\ \epsilon_0+\delta_1]$ such that $\sigma'(\epsilon_0) = \sigma_0$, $z(\epsilon_0) = 0$ and $G(\sigma'(\epsilon),\epsilon,z(\epsilon) \equiv 0$. The continuous vector $U_2(\epsilon) = \begin{pmatrix} z(\epsilon) \\ 1 \end{pmatrix}$ is therefore an eigenvector with eigenvalue 1 of $\mathbf{B}(\sigma'(\epsilon),\epsilon)$. Moreover, $U_1(\epsilon)$ and $U_2(\epsilon)$ are independent for ϵ close to ϵ_0. By the uniqueness of the spectral value established above, we get $\sigma'(\epsilon) = \sigma(\epsilon)$. This means that $\mathbf{B}(\sigma(\epsilon),\epsilon) \equiv \mathbf{I}$ for ϵ close to ϵ_0. From this and (2.5) we get that

$$\det(\mathbf{B}(\sigma(\epsilon),\epsilon)) = \exp\left(\frac{4\pi b\epsilon}{|\lambda_\epsilon|^2}\sigma(\epsilon)\right) \equiv 1$$

and therefore, $\sigma(\epsilon) = i\dfrac{|\lambda_\epsilon|^2}{2b\epsilon}k$ for every ϵ. Since $\mathbf{B}$ is analytic, we get that $\mathbf{B}(\sigma(\epsilon),\epsilon) = \mathbf{I}$ for every $\epsilon \in \mathbb{R}$. Now if we go back to the system $(\mathrm{E}_{\sigma(\epsilon),\epsilon})$, we can assume, by continuity and Corollary 2.0.1, that for the solution (ϕ,ψ), the function ϕ is dominating for every ϵ, i.e. $|\phi(t,\epsilon)| > |\psi(t,\epsilon)|$. The winding number $j_0 = \mathrm{Ind}(\phi)$ is then constant and we get from the first equation of $(\mathrm{E}_{\sigma(\epsilon),\epsilon})$ that

$$\frac{1}{2\pi}\lambda_\epsilon\int_0^{2\pi}\frac{\phi'(t,\epsilon)}{\phi(t,\epsilon)}dt = \lambda_\epsilon j_0 = (\sigma(\epsilon) - \lambda_\epsilon\nu) + \frac{1}{2\pi i}\int_0^{2\pi} c(t)\frac{\psi(t,\epsilon)}{\phi(t,\epsilon)}dt$$

By taking the limit, we obtain

$$\lim_{\epsilon\to 0}\sigma(\epsilon) = \lim_{\epsilon\to 0} i\frac{|\lambda_\epsilon|^2}{2b\epsilon}k = \lim_{\epsilon\to 0}\left(\lambda_\epsilon(j_0+\nu) + \frac{1}{2\pi i}\int_0^{2\pi} c(t)\frac{\psi(t,\epsilon)}{\phi(t,\epsilon)}dt\right)$$

Since the right hand side is bounded and $\lambda_\epsilon = a + ib\epsilon$ with $a > 0$ and $b \neq 0$, then necessarily $k = 0$ and this is a contradiction. This proves that $\mathbf{B}(\sigma_0, \epsilon_0) \neq \mathbf{I}$ $\square$

The following corollary is a direct consequence of Proposition 2.11 and formula (2.5)

Corollary 2.0.5 *If 1 is an eigenvalue of the monodromy matrix* $\mathbf{B}(\sigma,\epsilon)$ *with* $\epsilon \neq 0$, *then it has multiplicity one*

2.9 Main result about basic solutions

The following theorem summarizes the main properties of the basic solutions of $\mathcal{L}_\epsilon$.

Theorem 2.1 *For every* $j \in \mathbb{Z}$ *there are exactly two* $\mathbb{R}$*-independent basic solutions* $w_j^+(r,t,\epsilon)$ *and* $w_j^-(r,t,\epsilon)$ *of* $\mathcal{L}_\epsilon$ *with* $\mathrm{Char}(w_j^\pm) = (\sigma_j^\pm, j)$ *such that the spectral values* $\sigma_j^\pm \in \mathrm{Spec}(\mathcal{L}_\epsilon)$ *satisfy*

- $\sigma_j^\pm(\epsilon)$ *depends continuously on* ϵ *and*
- *if for some* $\epsilon_0 \in \mathbb{R}$, $\sigma_j^+(\epsilon_0) \in \mathbb{C}\backslash\mathbb{R}$, *then* $\sigma_j^-(\epsilon_0) = \sigma_j^+(\epsilon_0)$

Proof. Consider the analytic variety $\Gamma = \{(\sigma,\epsilon) \in \mathbb{C}\times\mathbb{R};\ F(\sigma,\epsilon) = 0\}$ where F is the spectral function given in (2.6). Thus the real spectral values $\tau_k^\pm$ of $\mathcal{L}_0$ (see Proposition 2.3) satisfy $(\tau_k^-, 0)$ and $(\tau_k^+, 0) \in \Gamma$. Let $\Gamma_j^\pm$ be connected components of Γ containing $(\tau_j^\pm, 0)$ and $\Gamma_j = \Gamma_j^- \cup \Gamma_j^+$.

For $j \neq k$ we have $\Gamma_k \cap \Gamma_j = \emptyset$. Indeed, if there is $(\sigma_0,\epsilon_0) \in \Gamma_k \cap \Gamma_j$ (with $\epsilon_0 \neq 0$), then equation $(\mathrm{E}_{\sigma_0,\epsilon_0})$ would have periodic solutions,

$$\begin{pmatrix} \phi_j(t,\sigma_0,\epsilon_0) \\ \psi_j(t,\sigma_0,\epsilon_0) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \phi_k(t,\sigma_0,\epsilon_0) \\ \psi_k(t,\sigma_0,\epsilon_0) \end{pmatrix}$$

giving rise to basic solutions

$$w_j = r^{\sigma_0}\phi_j + \overline{r^{\sigma_0}\psi_j} \quad \text{and} \quad w_k = r^{\sigma_0}\phi_k + \overline{r^{\sigma_0}\psi_k}$$

with winding numbers j and k, respectively. But the monodromy matrix $\mathbf{B}(\sigma_0,\epsilon_0)$ has only one eigenvector with eigenvalue 1 (Corollary 2.0.5). This means $\phi_k = c\phi_j$ and $\psi_k = c\psi_j$ for some constant c. Hence, $w_k = r^{\sigma_0}c\phi_j + \overline{r^{\sigma_0}c\psi_j}$ has also winding number j. This is a contradiction.

To complete the proof, we need to show that for every $j \in \mathbb{Z}$ and for every $\epsilon_0 \in \mathbb{R}$

$$|\Gamma_j \cap \{(\sigma, \epsilon_0);\ \sigma \in \mathbb{C}\}| \leq 2\,,$$

where $|S|$ denotes the cardinality of the set S. By contradiction, suppose that there exists $j_0 \in \mathbb{Z}$ and $\epsilon_0 \in \mathbb{R}^*$ such that

$$|\Gamma_{j_0} \cap \{(\sigma, \epsilon_0);\ \sigma \in \mathbb{C}\}| \geq 3\,.$$

Let (σ_1, ϵ_0), (σ_2, ϵ_0), and (σ_3, ϵ_0) be three distinct points in Γ_{j_0}. Hence, Γ_{j_0} has three distinct components C_1, C_2, and C_3 over a neighborhood of ϵ_0. They are defined by functions $f_1(\epsilon)$, $f_2(\epsilon)$ and $f_3(\epsilon)$, that are analytic, except possibly at ϵ_0. By analytic continuation, Γ_{j_0} has three distinct branches C_1, C_2, C_3, parametrized by ϵ. That is $C_l = \{(f_l(\epsilon), \epsilon);\ \epsilon \in \mathbb{R}\}$ with $f_l \in C^0(\mathbb{R})$ and analytic everywhere, except on a set of isolated points. In particular for $\epsilon = 0$, we get

$$\{f_1(0),\ f_2(0),\ f_3(0)\} = \{\tau_{j_0}^-,\ \tau_{j_0}^+\}$$

In the case $\tau_{j_0}^- < \tau_{j_0}^+$, we can assume that $f_1(0) = f_2(0)$ and this contradicts Proposition 2.8. In the case $\tau_{j_0}^- = \tau_{j_0}^+$, we would have $f_1(0) = f_2(0) = f_3(0) = \tau_{j_0}^+$ and this would contradict Proposition 2.10 $\square$

For the adjoint operator we have the following theorem.

Theorem 2.2 *Let*

$$w_\epsilon(r,t) = r^{\sigma(\epsilon)}\phi(t,\epsilon) + \overline{r^{\sigma(\epsilon)}\psi(t,\epsilon)}$$

be a basic solution of $\mathcal{L}_\epsilon$ with $\mathrm{Char}(w_\epsilon) = (\sigma(\epsilon), j)$. *Then $\mathcal{L}_\epsilon^*$ has a basic solution*

$$w_\epsilon^\star(r,t) = r^{-\sigma(\epsilon)}X(t,\epsilon) + \overline{r^{-\sigma(\epsilon)}Z(t,\epsilon)}$$

with $\mathrm{Char}(w_\epsilon^\star) = (-\sigma(\epsilon), -j)$.

Proof. For $\sigma(\epsilon) \in \mathrm{Spec}(\mathbf{B}(.,\epsilon))$, it follows from (2.17) and (2.16) that $\sigma(-\epsilon) = \sigma(\epsilon)$ and $\overline{\sigma(\epsilon)} = \overline{\sigma(-\epsilon)} \in \mathrm{Spec}(\mathbf{B}(.,-\epsilon))$. Let $U(\epsilon)$ be a continuous eigenvector with eigenvalue 1 of $\mathbf{B}(\overline{\sigma(-\epsilon)}, -\epsilon)$ such that

$$U(0) = \begin{pmatrix} \phi(0,0) \\ \psi(0,0) \end{pmatrix}\,.$$

Then $\mathbf{V}(t, \overline{\sigma(-\epsilon)}, -\epsilon)U(\epsilon)$ is a periodic solution of $(\mathrm{E}_{\overline{\sigma(-\epsilon)},-\epsilon})$. The function

$$\begin{pmatrix} X(t,\epsilon) \\ Z(t,\epsilon) \end{pmatrix} = \mathbf{D}\overline{\mathbf{V}(t, \overline{\sigma(-\epsilon)}, -\epsilon)U(\epsilon)} = \widetilde{\mathbf{V}}(t, -\sigma(\epsilon), \epsilon)\mathbf{D}\overline{U(\epsilon)}$$

is a periodic solution of the adjoint system $(\widetilde{\mathrm{E}}_{-\sigma(\epsilon),\epsilon})$. Furthermore, $|X| > |Z|$ and $\mathrm{Ind}(X) = -j$ so that the character of the associated basic solution is $(-\sigma(\epsilon), -j)$ $\square$

3 Example

We give here an example in which the basic solutions can be explicitly determined. This is the case when $c(t) = ic_0 \mathrm{e}^{ikt}$ with $c_0 \in \mathbb{C}^*$. For simplicity, we assume that $\nu = 0$. The system of equations $(\mathrm{E}_{\sigma,\epsilon})$ is

$$\begin{aligned} \lambda_\epsilon \dot{\phi}(t) &= i\sigma\phi(t) + ic_0 \mathrm{e}^{ikt}\psi(t) \\ \overline{\lambda_\epsilon}\dot{\psi}(t) &= -i\sigma\psi(t) - i\overline{c_0}\mathrm{e}^{-ikt}\phi(t) \end{aligned}$$

In this case we can use Fourier series to determine the spectral values and the periodic solutions. For a given $j \in \mathbb{Z}$, the system has a solution of the form $\phi(t) = \mathrm{e}^{ijt}$, $\psi(t) = D\mathrm{e}^{i(j-k)t}$ with σ and D satisfying

$$\begin{aligned} &\lambda_\epsilon j = \sigma + c_0 D \\ &\overline{\lambda_\epsilon}(j-k)D = -\sigma D - \overline{c_0} \end{aligned}$$

The elimination of D, gives the following quadratic equation for the spectral value σ

$$\sigma^2 - [(\lambda_\epsilon - \overline{\lambda_\epsilon})j + k\overline{\lambda_\epsilon}]\sigma - [j(j-k)|\lambda_\epsilon|^2 + |c_0|^2] = 0$$

After replacing λ_ϵ by $a + ib\epsilon$ we get σ and D:

$$\begin{aligned} \sigma_j &= ib\epsilon j + \frac{(a-ib\epsilon)k}{2} + \sqrt{\left(aj - \frac{(a-ib\epsilon)k}{2}\right)^2 + |c_0|^2} \\ D_j &= \frac{(a+ib\epsilon)j - \sigma_j}{c_0} \end{aligned}$$

The corresponding basic solution of $\mathcal{L}_\epsilon$ is

$$w_j(r,t,\epsilon) = r^{\sigma_j}\mathrm{e}^{ijt} + \overline{r^{\sigma_j} D_j \mathrm{e}^{i(j-k)t}}\,.$$

The character of w_j is (σ_j, j) if $|D_j| < 1$ and it is $(\overline{\sigma_j}, k-j)$ if $|D_j| > 1$.

Note that in order for D_j to have norm 1, for some j_0, say $D_{j_0} = \mathrm{e}^{i\alpha}$, the exponent σ_{j_0} needs to satisfy

$$\sigma_{j_0} = \lambda_\epsilon j_0 - c_0\mathrm{e}^{i\alpha}, \quad \text{and} \quad \sigma_{j_0} = -\overline{\lambda_\epsilon}(j_0 - k) - \overline{c_0}\mathrm{e}^{-i\alpha}\,.$$

Consequently, $\lambda_\epsilon(2j_0 - k) = \sigma_{j_0} - \overline{\sigma_{j_0}}$. Since $\mathrm{Re}(\lambda_\epsilon) = a > 0$, then necessarily $k = 2j_0$ is an even integer.

Thus for k odd, $|D_j| \neq 1$ for every $j \in \mathbb{Z}$ and for an even k, $k = 2j_0$, we have $|D_j| \neq 1$ for every $j \neq j_0$. At the level j_0, we get

$$\sigma_{j_0} = aj_0 + \sqrt{-b^2\epsilon^2 j_0^2 + |c_0|^2} \quad \text{and} \quad D_{j_0} = \frac{ib\epsilon j_0 - \sqrt{-b^2\epsilon^2 j_0^2 + |c_0|^2}}{c_0}$$

and the character of the corresponding basic solution is (σ_{j_0}, j_0).

4 Asymptotic behavior of the basic solutions of $\mathcal{L}$

In this section, we determine the asymptotic behavior of the basic solutions. This behavior will be needed in the next section to estimate the kernels. From now on, there is no need anymore for the parameter ϵ. So we will denote $\mathcal{L}_1$ by $\mathcal{L}$ and the associated system of differential equations $(\mathrm{E}_{\sigma,1})$ by (E_σ). We will assume here that $\lambda = a + ib$ with $a > 0$ and $b \neq 0$, since the asymptotic behavior in case $b = 0$ is known from [9]. Hence,

$$\mathcal{L}u = Lu + i\lambda\nu u - c(t)\overline{u}\,,$$

where L is the vector field given in (1.1). Let

$$\gamma = \frac{1}{4a\pi}\int_0^{2\pi} |c(t)|^2 dt \quad \text{and} \quad k(t) = \frac{1}{\lambda}\left[\gamma t - \frac{1}{2a}\int_0^t |c(s)|^2 ds\right] \tag{4.1}$$

Note that $k(t)$ is 2π-periodic. We have the following theorem

Theorem 4.1 *For $j \in \mathbb{Z}$, the operator $\mathcal{L}$ has basic solution*

$$w_j(r,t) = r^{\sigma_j}\phi_j(t) + \overline{r^{\sigma_j}\psi_j(t)}$$

with character (σ_j, j) such that, as $|j| \to \infty$, we have

$$\sigma_j = \lambda(j+\nu) + \frac{\gamma}{j} + O(j^{-2}) \tag{4.2}$$

$$\phi_j(t) = \mathrm{e}^{ijt}\left(1 + i\frac{k(t)}{j}\right) + O(j^{-2}) \tag{4.3}$$

$$\psi_j(t) = -\mathrm{e}^{ijt}\frac{\overline{c(t)}}{2aj} + O(j^{-2}) \tag{4.4}$$

where γ and $k(t)$ are given in (4.1). *Furthermore, any basic solution (with $|j|$ large) has the form*

$$w(r,t) = r^{\sigma_j}a\phi_j(t) + \overline{r^{\sigma_j}a\psi_j(t)},$$

with $a \in \mathbb{C}$

Remark 4.1 It follows from Theorems 2.2 and 4.1 that for $|j| \in \mathbb{Z}$ large, the basic solutions of $\mathcal{L}^*$ have the form,

$$w^*(r,t) = r^{-\sigma_j}X_{-j}(t) + \overline{r^{-\sigma_j}Z_{-j}(t)}$$

with $\mathrm{Char}(w^*) = (-\sigma_j, -j)$ where

$$\begin{aligned} X_{-j}(t) &= \mathrm{e}^{-ijt}\left(1 - i\frac{k(t)}{j}\right) + O(j^{-2}) \\ Z_{-j}(t) &= -\mathrm{e}^{-ijt}\frac{c(t)}{2aj} + O(j^{-2}) \end{aligned}$$

The remainder of this section deals with the proof of Theorem 4.1. The proof will be divided into 3 steps. To simplify the expressions, we will use the following variables

$$\mu = \frac{\sigma - \lambda\nu}{\lambda}, \ \mathrm{e}^{ix} = \lambda/\overline{\lambda}, \ \delta = (\mathrm{e}^{-ix} + 1)\nu, \ \text{and } c_1(t) = \frac{c(t)}{\lambda} \tag{4.5}$$

The system of equations (E_σ) becomes then

$$\begin{aligned} \dot{\phi} &= i\mu\phi + c_1(t)\psi \\ \dot{\psi} &= -i\mathrm{e}^{ix}(\mu + \delta)\psi + \overline{c_1(t)}\phi \end{aligned} \tag{4.6}$$

Now, we proceed with the proof of the theorem.

4.1 Estimate of σ

For a periodic solution (ϕ, ψ) of (4.6) with $|\psi| < |\phi|$, we can assume that $\max|\phi| = 1$ and $\mathrm{Ind}(\phi) = j$. Let

$$T(t) = \frac{\psi(t)}{\phi(t)} \tag{4.7}$$

It follows from the first equation of (4.6) that

$$\mu + \frac{1}{2\pi i}\int_0^{2\pi} c_1(t)T(t)dt = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\dot{\phi}(t)}{\phi(t)}dt = j$$

Hence,

$$\mu = j - M_j \quad \text{with} \quad M_j = \frac{1}{2\pi i}\int_0^{2\pi} c_1(t)T(t)dt\,. \tag{4.8}$$

Note that $|M_j| \leq \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |c_1(t)|dt$. To obtain a better estimate of M_j we use the second equation of (4.6) to get (after multiplying by c_1, dividing by ϕ and integrating over $[0,\ 2\pi]$)

$$-\,i\mathrm{e}^{ix}(\mu + \delta)M_j = \frac{-1}{2\pi i}\int_0^{2\pi} |c_1(t)|^2 dt + \frac{1}{2\pi i}\int_0^{2\pi} c_1(t)\frac{\psi'(t)}{\phi(t)}dt \tag{4.9}$$

We use integration by parts in the last integral together with (4.6) to obtain

$$\begin{aligned} \int_0^{2\pi} c_1(t)\frac{\psi'(t)}{\phi(t)}dt &= -\int_0^{2\pi} c_1'(t)T(t)dt + \int_0^{2\pi} c_1(t)\frac{(i\mu\phi(t) + c_1(t)\psi(t))\psi(t)}{\phi(t)^2}dt \\ &= \int_0^{2\pi} (-c_1'(t)T(t) + i\mu c_1(t)T(t) + c_1(t)^2T(t)^2)dt \end{aligned}$$

From this and (4.9), we deduce that

$$-i\left[(1+\mathrm{e}^{ix})\mu+\delta\mathrm{e}^{ix}\right]M_j=\frac{-1}{2\pi i}\int_0^{2\pi}(|c_1(t)|^2+c_1'(t)T(t)-c_1^2(t)T^2(t))dt.$$

Consequently,

$$|(1+\mathrm{e}^{ix})\mu+\delta\mathrm{e}^{ix}|\,|M_j|\le A_1\,, \tag{4.10}$$

where $A_1=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}(2|c_1(t)|^2+|c_1'(t)|)dt$. Note that since $1+\mathrm{e}^{ix}=2a/\overline{\lambda}$ satisfies $\mathrm{Re}(1+\mathrm{e}^{ix})>0$ and $|\delta|<2$, then it follows from (4.8) and the boundedness of M_j that there exists $J_0\in\mathbb{Z}^+$ such that

$$|(1+\mathrm{e}^{ix})\mu+\delta\mathrm{e}^{ix}|\ge\frac{|j|}{2},\qquad\forall j\in\mathbb{Z},\;\;|j|\ge J_0. \tag{4.11}$$

Lemma 4.1 *Let $N_j=jM_j$, then*

$$\lim_{|j|\to\infty}N_j=\frac{-1}{2\pi(1+\mathrm{e}^{ix})}\int_0^{2\pi}|c_1(t)|^2dt=-\frac{\gamma}{\lambda} \tag{4.12}$$

where γ is given in (4.1)

Proof. It follows from (4.10) and (4.11) that $|N_j|\le 2A_1$. Let

$$\phi(t)=\mathrm{e}^{ijt}\phi_1(t)\quad\text{and}\quad\psi(t)=\mathrm{e}^{ijt}\psi_1(t) \tag{4.13}$$

Hence, $\mathrm{Ind}(\phi_1)=0$ and $T=\psi_1/\phi_1$. From (4.6), we get the system for $\phi_1,\ \psi_1$

$$\begin{aligned}\dot{\phi}_1&=-i\frac{N_j}{j}\phi_1+c_1(t)\psi_1\\ \dot{\psi}_1&=-iE_j\psi_1+\overline{c_1(t)}\phi_1\end{aligned} \tag{4.14}$$

where

$$E_j=(1+\mathrm{e}^{ix})j-\mathrm{e}^{ix}\frac{N_j}{j}+\mathrm{e}^{ix}\delta\,. \tag{4.15}$$

Note that since $|N_j|\le 2A_1$ and $\mathrm{e}^{ix}\ne-1$, then for $|j|$ large ($|j|\ge J_0$), we have

$$|E_j|\ge|1+\mathrm{e}^{ix}|\frac{|j|}{2}\,. \tag{4.16}$$

Now, it follows from the first equation of (4.14) and from $\mathrm{Ind}(\phi_1)=0$ that

$$\frac{N_j}{j}=\frac{1}{2\pi i}\int_0^{2\pi}c_1(t)T(t)dt\,. \tag{4.17}$$

We use the second equation of (4.14) to estimate the integral appearing in (4.17)

$$\int_0^{2\pi}c_1Tdt=\frac{-1}{iE_j}\int_0^{2\pi}c_1\frac{\psi_1'-\overline{c_1}\phi_1}{\phi_1}dt=\frac{1}{iE_j}\int_0^{2\pi}\left(|c_1|^2dt-c_1\frac{\psi_1'}{\phi_1}\right)dt \tag{4.18}$$

We use integration by parts and system (4.14) to evaluate the last integral appearing in (4.18).

$$\begin{aligned}\int_0^{2\pi} c_1\frac{\psi_1'}{\phi_1}dt &= -\int_0^{2\pi}\left[c_1'T + c_1\psi_1\frac{(-iN_j/j)\phi_1 + c_1\psi_1}{\phi_1^2}\right]dt \\ &= \int_0^{2\pi}(-c_1'T + c_1^2T^2)dt - \frac{iN_j}{j}\int_0^{2\pi} c_1 T dt\end{aligned} \tag{4.19}$$

Thus,

$$\int_0^{2\pi} c_1 T dt = \frac{1}{iE_j}\int_0^{2\pi}(|c_1|^2 + c_1'T - c_1^2T^2)dt + \frac{N_j}{jE_j}\int_0^{2\pi} c_1 T dt \tag{4.20}$$

Therefore, from (4.20) and (4.16), we have

$$\int_0^{2\pi} c_1 T dt = \frac{1}{iE_j}\left[\int_0^{2\pi}|c_1|^2 dt + I_1 - I_2\right] + O(\frac{1}{j^2}) \tag{4.21}$$

where I_1 and I_2 are given by

$$I_1 = \int_0^{2\pi} c_1'(t)T(t)dt \quad \text{and} \quad I_2 = \int_0^{2\pi} c_1^2(t)T^2(t)dt.$$

Now we show that $I_1 = O(1/j)$ and $I_2 = O(1/j)$. For I_1, we have, after using system (4.14), that

$$\begin{aligned} I_1 &= \frac{-1}{iE_j}\int_0^{2\pi} c_1'\frac{\psi_1' - \overline{c_1}\phi_1}{\phi_1}dt \\ &= \frac{-1}{iE_j}\left[\int_0^{2\pi}|c_1|^2 dt + \int_0^{2\pi} c_1''T dt - \int_0^{2\pi} c_1'\psi_1\frac{\phi_1'}{\phi_1^2}dt\right] \\ &= \frac{-1}{iE_j}\left[\int_0^{2\pi}(|c_1|^2 + c_1''T)dt - \int_0^{2\pi} c_1'\psi_1\frac{(-iN_j/j)\phi_1 + c_1\psi_1}{\phi_1^2}dt\right]\end{aligned}$$

So

$$iE_jI_1 = \int_0^{2\pi}(|c_1|^2 + c_1''T - c_1'c_1T^2)dt + \frac{iN_j}{j}\int_0^{2\pi} c_1'T dt \tag{4.22}$$

Since $|T| < 1$ and $|N_j| < 2A_1$, we get $I_1 = O(1/j)$. For I_2, we use system (4.14) and integration by parts, to obtain

$$\begin{aligned} -iE_jI_2 &= \int_0^{2\pi} c_1^2\frac{\psi_1}{\phi_1^2}(\psi_1' - \overline{c_1}\phi_1)dt = -\int_0^{2\pi} c_1^2\overline{c_1}T dt + \frac{1}{2}\int_0^{2\pi} c_1^2\frac{(\psi_1^2)'}{\phi_1^2}dt \\ &= -\int_0^{2\pi} c_1^2\overline{c_1}T dt - \int_0^{2\pi} c_1c_1'T^2 dt + \int_0^{2\pi} c_1^2\psi_1^2\frac{\phi_1'}{\phi_1^3}dt \\ &= -\int_0^{2\pi}(c_1^2\overline{c_1}T + c_1c_1'T^2)dt + \int_0^{2\pi} c_1^2\psi_1^2\frac{(-iN_j/j)\phi_1 + c_1\psi_1}{\phi_1^3}dt \\ &= -\int_0^{2\pi}(c_1^2\overline{c_1}T + c_1c_1'T^2 - c_1^3T^3)dt - \frac{iN_j}{j}\int_0^{2\pi} c_1^2T^2 dt\end{aligned} \tag{4.23}$$

and again $|E_j I_2|$ is bounded and therefore $I_2 = O(1/j)$. With these estimates for I_1 and I_2, expressions (4.17), (4.20), and (4.21) give

$$N_j = -\frac{j}{2\pi E_j}\left[\int_0^{2\pi} |c_1|^2 dt + O(|j|^{-1})\right] .$$

Since $\lim_{|j|\to\infty} \frac{j}{E_j} = \frac{1}{1+\mathrm{e}^{ix}}$, the lemma follows $\square$

Using this lemma we get

$$M_j = \frac{N_j}{j} = \frac{-\overline{\lambda}}{2\pi j(\lambda+\overline{\lambda})}\int_0^{2\pi} \frac{|c(t)|^2}{|\lambda|^2} dt + O(j^{-2}) = -\frac{\gamma}{\lambda j} + O(j^{-2}).$$

Consequently, $\mu = j - M_j = j + \frac{\gamma}{j\lambda} + O(j^{-2})$ and $\sigma = \lambda(\mu+\nu)$ gives estimate (4.2) of the theorem.

4.2 First estimate of ϕ and ψ

We begin with the estimates of the components ϕ and ψ of the basic solutions. Let ϕ_1, and ψ_1 be the functions defined in (4.13). Note that $\max|\phi_1| \le 1$ and $|\psi_1| < |\phi_1|$. We have the following lemma that gives an estimate of ψ_1.

Lemma 4.2 *There exist $J_0 \in \mathbb{Z}^+$ and $K > 0$ such that the function $\psi_2 = j\psi_1$ satisfies*

$$|\psi_2(t)| = |j\psi_1(t)| \le K, \qquad \forall t \in \mathbb{R},\ \forall j \in \mathbb{Z},\ \ |j| \ge J_0 . \tag{4.24}$$

Proof. The system (4.14) implies that $T = \psi/\phi = \psi_1/\phi_1$ satisfies the equation

$$T'(t) = -L_j T(t) - c_1(t)T^2(t) + \overline{c_1}(t) \tag{4.25}$$

where

$$L_j = i\left(E_j - \frac{N_j}{j}\right) = i\left[(1+\mathrm{e}^{ix})j + \mathrm{e}^{ix}\delta + O(1/j)\right] . \tag{4.26}$$

Note that the real part, p, of L_j is given by

$$p = \mathrm{Re}(L_j) = -j\sin x + \mathrm{Re}(\delta \mathrm{e}^{ix}) + O(1/j) .$$

and, since $\sin x \ne 0$ (because $b \ne 0$), there exists $J_0 \in \mathbb{Z}^+$ such that

$$|p| \ge \frac{|\sin x|}{2}|j| \qquad \forall j \in \mathbb{Z},\ |j| \ge J_0 \tag{4.27}$$

Let $\rho(t) = |T(t)|$, $A(t) = \arg T(t)$, $\vartheta(t) = \arg c_1(t)$ and $M = \max_{0\le t\le 2\pi} |c_1(t)|$. Let us rewrite equation (4.25) as

$$\rho' + iA'\rho = -L_j\rho - |c_1|\rho^2 \mathrm{e}^{i(A+\vartheta)} + |c_1|\mathrm{e}^{-i(A+\vartheta)}. \tag{4.28}$$

By taking the real part, we obtain

$$\rho' = -p\rho - |c_1|\rho^2 \cos(A+\vartheta) + |c_1|\cos(A+\vartheta). \tag{4.29}$$

Since $0 \le \rho < 1$, we get $-2M \le \rho' + p\rho \le 2M$. Equivalently,

$$-2M\mathrm{e}^{pt} \le (\rho(t)\mathrm{e}^{pt})' \le 2M\mathrm{e}^{pt}. \tag{4.30}$$

In the case where $p > 0$ (and so $p > |j\sin x|/2$), we obtain, after integrating (4.30) from 0 to t, with $t > 0$, that

$$\rho(t) \le \left(\rho(0) - \frac{2M}{p}\right)\mathrm{e}^{-pt} + \frac{2M}{p}. \tag{4.31}$$

Let $t_0 > 0$ be such that $\mathrm{e}^{-pt_0} \le 2M/p$. Then, it follows from (4.31), that

$$0 \le \rho(t) \le \frac{2M}{p}\left[1 + \rho(0) - \frac{2M}{p}\right] \le \frac{K}{j}, \qquad \forall t \ge t_0 \tag{4.32}$$

where K is a constant independent on j. Since the function ρ is periodic, then inequality (4.32) holds for every $t \in \mathbb{R}$. When $p < 0$, an analogous argument (integrating (4.30) from t to 0 with $t < 0$) yields the estimate (4.32). Hence, $|T| < K/j$ and this completes the proof of the lemma $\square$

Throughout the remainder of this section, we will use Fourier series. For a function $F \in L^2(\mathbb{S}^1, \mathbb{C})$, we denote by $F^{(l)}$ its l-th Fourier coefficient:

$$F^{(l)} = \frac{1}{2\pi i}\int_0^{2\pi} F(t)\mathrm{e}^{-ilt}dt$$

An estimate of ϕ_1 is given in the following lemma.

Lemma 4.3 *There exist $J_0 \in \mathbb{Z}^+$ and positive constants K_1, K_2 such that for $j \in \mathbb{Z}$, $|j| > J_0$, the function ϕ_1 has the form*

$$\phi_1(t) = \phi_1^{(0)} + \frac{\phi_2(t)}{j} \tag{4.33}$$

with $|\phi_2(t)| \le K_1$ for every $t \in \mathbb{R}$, $\phi_2^{(0)} = 0$, and $\frac{1}{2} \le |\phi_1^{(0)}| \le K_2$.

Proof. We estimate the Fourier coefficients of ϕ_1 in terms of those of the differentiable function $c_1\psi_2$, where ψ_2 is the function in Lemma 4.2. For this, we replace ψ_1 by ψ_2/j in (4.14) and rewrite the first equation as

$$\phi_1'(t) = -i\frac{N_j}{j}\phi_1(t) + c_1(t)\frac{\psi_2(t)}{j}\,. \tag{4.34}$$

In terms of the Fourier coefficients, we get then

$$i\left(l+\frac{N_j}{j}\right)\phi_1^{(l)}=\frac{1}{j}(c_1\psi_2)^{(l)}\qquad l\in\mathbb{Z}\tag{4.35}$$

Since $N_j\sim-\dfrac{\gamma}{\lambda}$ (by Lemma 4.1), then there exists $J_0\in\mathbb{Z}^+$ such that

$$\left|\mathrm{Re}\left(l+\frac{N_j}{j}\right)\right|\geq\frac{|l|}{2}\qquad\forall l\in\mathbb{Z}^*$$

It follows from (4.35) that

$$|\phi_1^{(l)}|\leq\frac{2}{|j|}\left(\frac{|(c_1\psi_2)^{(l)}|}{|l|}\right)\qquad\forall l\in\mathbb{Z}^*\tag{4.36}$$

The function $\phi_2(t)=j(\phi_1(t)-\phi_1^{(0)})$ satisfies the conditions of the lemma. For $l=0$, we get

$$\phi_1^{(0)}=\frac{-i}{N_j}(c_1\psi_2)^{(0)}$$

and since ψ_2 is bounded, then $|\phi_1^{(0)}|\leq K_2$. Finally, $|\phi_1^{(0)}|\geq 1/2$ for $|j|$ large since $|\phi_2|\leq K_1$ and $\max|\phi_1|=1$ $\square$

4.3 End of the proof of Theorem 4.1

Now we estimate the functions ψ_2 and ϕ_2 that are defined in the previous lemmas

Lemma 4.4 *There exist $J_0\in\mathbb{Z}^+$ and $K>0$ such that for $j\in\mathbb{Z}$, $|j|\geq J_0$, the function ψ_2, given in Lemma 4.2, has the form*

$$\psi_2(t)=\phi_1^{(0)}\frac{\overline{c_1}(t)}{i(1+\mathrm{e}^{ix})}+\frac{\psi_4(t)}{j}\tag{4.37}$$

with ψ_4 satisfying $|\psi_4(t)|\leq K$ for every $t\in\mathbb{R}$

Proof. With ϕ_2 and ψ_2 as in Lemmas 4.2 and 4.3, we rewrite the system (4.14) and obtain

$$\begin{aligned}\phi_2'(t)&=-iN_j\left(\phi_1^{(0)}+\frac{\phi_2(t)}{j}\right)+c_1(t)\psi_2(t)\\ \psi_2'(t)&=-iE_j\psi_2(t)+\overline{c_1}(t)(j\phi_1^{(0)}+\phi_2(t))\end{aligned}\tag{4.38}$$

From (4.38), we see that the Fourier coefficients satisfy

$$\psi_2^{(l)}=\phi_1^{(0)}\frac{j}{i(l+E_j)}\overline{c_1}^{(l)}+\frac{(\overline{c_1}\phi_2)^{(l)}}{i(l+E_j)},\qquad l\in\mathbb{Z}\tag{4.39}$$

Let

$$\psi_3(t) = \psi_2(t) - \phi_1^{(0)} \frac{\overline{c_1}(t)}{i(1+\mathrm{e}^{ix})}. \tag{4.40}$$

It follows at once from (4.39) that the Fourier coefficients of ψ_3 satisfy

$$\psi_3^{(l)} = \phi_1^{(0)} \frac{(1+\mathrm{e}^{ix})j - (l+E_j)}{i(l+E_j)(1+\mathrm{e}^{ix})} \overline{c_1}^{(l)} + \frac{(\overline{c_1}\phi_2)^{(l)}}{i(l+E_j)}, \qquad l \in \mathbb{Z} \tag{4.41}$$

Since for $|j|$ large, we have

$$|(1+\mathrm{e}^{ix})j - E_j| = |\frac{N_j}{j} - \delta| < 3 \quad \text{and} \quad |l+E_j| \geq |\mathrm{Im}(E_j)| \geq \frac{j}{C_1}$$

for some positive constant C_1, then from (4.41) we get

$$|\psi_3^{(l)}| \leq \frac{K}{j} \left(|\phi_1^{(0)} l \overline{c_1}^{(l)}| + |(\overline{c_1}\phi_2)^{(l)}| \right) \tag{4.42}$$

The term $il\overline{c_1}^{(l)}$ is the l-th Fourier coefficient of $\overline{c_1}' \in C^{k-1}$, and therefore it follows from (4.42) that $j\psi_3$ is uniformly bounded (in j). Thus the function $\psi_4 = \dfrac{\psi_3}{j} \in C^{k-1}$ is bounded $\square$

Lemma 4.5 *There exist $J_0 \in \mathbb{Z}^+$ and $K > 0$ such that for $j \in \mathbb{Z}$, $|j| \geq J_0$, the function ϕ_2, given in Lemma* 4.3*, has the form*

$$\phi_2(t) = i\phi_1^{(0)} k(t) + \frac{\phi_4(t)}{j} \tag{4.43}$$

where $k(t)$ is the function given in (4.1)*, and $\phi_4 \in C^k$ satisfies $|\phi_4(t)| \leq K$ for every $t \in \mathbb{R}$*

Proof. We use Lemma 4.1 to write

$$N_j = -\frac{\gamma}{\lambda} + \frac{C_j}{j}$$

with $C_j \in \mathbb{C}$ bounded. By using Lemmas 4.3 and 4.4 in (4.34), we obtain an equation for ϕ_2:

$$\phi_2'(t) = (i\frac{\gamma}{\lambda} - \frac{iC_j}{j})(\phi_1^{(0)} + \frac{\phi_2(t)}{j}) + \frac{\phi_1^{(0)}|c_1(t)|^2}{i(1+\mathrm{e}^{ix})} + \frac{c_1(t)\psi_4(t)}{j} \tag{4.44}$$

Equivalently,

$$\phi_2'(t) = i\phi_1^{(0)} \left[\frac{\gamma}{\lambda} - \frac{|c_1(t)|^2}{1+\mathrm{e}^{ix}} + \frac{C_j}{j\phi_1^{(0)}} \right] + \frac{1}{j} \left[i(\frac{\gamma}{\lambda} - \frac{C_j}{j})\phi_2(t) + c_1(t)\psi_4(t) \right] \tag{4.45}$$

Let $\phi_3(t) = \phi_2(t) - i\phi_1^{(0)}k(t)$, where $k(t)$ is the function defined in (4.1). Note that since $k'(t) = \dfrac{\gamma}{\lambda} - \dfrac{|c_1(t)|^2}{1+\mathrm{e}^{ix}}$, then it follows from (4.45) that

$$\phi_3'(t) = \frac{1}{j}\left[-iC_j + c_1(t)\psi_4(t) + i\left(\frac{\gamma}{\lambda} - \frac{C_j}{j}\right)(\phi_3(t) + i\phi_1^{(0)}k(t))\right] \qquad (4.46)$$

The Fourier coefficients of ϕ_3 satisfy

$$i\left[l - \frac{\gamma}{\lambda j} + \frac{C_j}{j^2}\right]\phi_3^{(l)} = \frac{1}{j}\left[-iC_j + (c_1\psi_4)^{(l)} - \left(\frac{\gamma}{\lambda} - \frac{C_j}{j}\right)\phi_1^{(0)}k^{(l)}\right] \qquad (4.47)$$

This shows that $j\phi_3$ is bounded $\quad\square$

From these lemmas, we get for the functions ϕ and ψ

$$\begin{aligned} \phi(t) &= \mathrm{e}^{ijt}\left(\phi_1^0 + i\phi_1^0\frac{k(t)}{j} + \frac{\phi_4(t)}{j^2}\right) \\ \psi(t) &= \mathrm{e}^{ijt}\left(\phi_1^0\frac{\overline{c_1}(t)}{ij(1+\mathrm{e}^{ix})} + \frac{\psi_4(t)}{j^2}\right) \end{aligned}$$

Since $\dfrac{1}{2} \leq \phi_1^{(0)} \leq K_2$, then we can divide the above functions by $\phi_1^{(0)}$ and obtain (4.3) and (4.4) of the Theorem $\square$

5 The kernels

We use the basic solutions to construct kernels for the operator $\mathcal{L}$. For $j, k \in \mathbb{Z}$, let

$$\begin{aligned} w_j^{\pm}(r,t) &= r^{\sigma_j^{\pm}}\phi_j^{\pm}(t) + \overline{r^{\sigma_j^{\pm}}\psi_j^{\pm}(t)} \\ w_k^{*\pm}(r,t) &= r^{\mu_k^{\pm}}X_k^{\pm}(t) + \overline{r^{\mu_k^{\pm}}Z_k^{\pm}(t)} \end{aligned} \qquad (5.1)$$

be the basic solutions of $\mathcal{L}$ and $\mathcal{L}^*$, respectively, with $\mathrm{Char}(w_j^{\pm}) = (\sigma_j^{\pm}, j)$, $\mathrm{Char}(w_k^{*\pm}) = (\mu_k^{\pm}, k)$ and such that $\phi_j^+(0) = 1$, $\phi_j^-(0) = i$, $X_k^+(0) = i$, and $X_k^-(0) = 1$. Note that, by Theorem 2.2, we have $\mu_k^{\pm} = -\sigma_{-j}^{\pm}$.

In the remainder of this paper we will use the following notation, $A^{\pm}B^{\pm} = A^+B^+ + A^-B^-$. Functions $f(r,t)$ and $g(\rho,\theta)$ will be denoted by $f(z)$ and $g(\zeta)$, where $z = r^{\lambda}\mathrm{e}^{it}$ and $\zeta = \rho^{\lambda}\mathrm{e}^{i\theta}$. Define functions $\Omega_1(z,\zeta)$ and $\Omega_2(z,\zeta)$ as follows:

$$\Omega_1(z,\zeta) = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{\mathrm{Re}(\sigma_j^{\pm})\geq 0} w_j^{\pm}(z)w_{-j}^{*\pm}(\zeta) & \text{if } r < \rho \\ -\dfrac{1}{2}\displaystyle\sum_{\mathrm{Re}(\sigma_j^{\pm})< 0} w_j^{\pm}(z)w_{-j}^{*\pm}(\zeta) & \text{if } r > \rho \end{cases} \qquad (5.2)$$

and

$$\Omega_2(z,\zeta) = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{\mathrm{Re}(\sigma_j^{\pm}) \geq 0} \overline{w_j^{\pm}(z)} w_{-j}^{*\pm}(\zeta) & \text{if } r < \rho \\ -\dfrac{1}{2} \displaystyle\sum_{\mathrm{Re}(\sigma_j^{\pm}) < 0} \overline{w_j^{\pm}(z)} w_{-j}^{*\pm}(\zeta) & \text{if } r > \rho \end{cases} \tag{5.3}$$

Let $K(t,\theta)$ and $L(z,\zeta)$ be defined by

$$K(t,\theta) = k(t) - k(\theta) \quad \text{and} \quad L(z,\zeta) = \begin{cases} \log \dfrac{\zeta}{\zeta - z} & \text{if } r < \rho \\ \log \dfrac{z}{z - \zeta} & \text{if } r > \rho \end{cases} \tag{5.4}$$

where $k(t)$ is the function defined in (4.1) and where log denotes the principal branch of the logarithm in $\mathbb{C}\backslash\mathbb{R}^-$. In the next theorem, we will use the notation

$$\Delta_1 = \{(r,t,\rho,\theta);\ 0 < r \leq \rho\}, \quad \Delta_2 = \{(r,t,\rho,\theta);\ 0 < \rho \leq r\}$$

and $\mathrm{Int}(\Delta_1)$, $\mathrm{Int}(\Delta_2)$ will denote their interiors.

Theorem 5.1 *The functions $C_1(z,\zeta)$ and $C_2(z,\zeta)$ defined in $\mathrm{Int}(\Delta_1) \cup \mathrm{Int}(\Delta_2)$ by*

$$\begin{aligned} C_1(z,\zeta) &= \Omega_1(z,\zeta) - i\left(\frac{r}{\rho}\right)^{\lambda\nu}\left[\frac{\zeta}{\zeta - z} + iK(t,\theta)L(z,\zeta)\right] \\ C_2(z,\zeta) &= \Omega_2(z,\zeta) - \frac{\overline{c(t)}}{2a}\left(\frac{r}{\rho}\right)^{\lambda\nu} L(z,\zeta) - \frac{\overline{c(\theta)}}{2a}\overline{\left(\frac{r}{\rho}\right)^{\lambda\nu} L(z,\zeta)} \end{aligned} \tag{5.5}$$

are in $C^1(\Delta_1) \cup C^1(\Delta_2)$, meaning that the restrictions of $C_{1,2}$ to $\mathrm{Int}(\Delta_1)$ (or to $\mathrm{Int}(\Delta_2)$) extend as C^1 functions to Δ_1 (or Δ_2). Furthermore, for any $R > 0$, the functions $C_{1,2}$ are bounded for $r \leq R$ and $\rho \leq R$

To prove this theorem, we need two lemmas.

5.1 Two lemmas

Lemma 5.1 *For $|j|$ large, we have*

$$\begin{aligned} w_j^{\pm}(z) w_{-j}^{*\pm}(\zeta) &= 2i\left(\frac{r}{\rho}\right)^{\sigma_j} e^{ij(t-\theta)}\left(1 + i\frac{K(t,\theta)}{j} + O(j^{-2})\right) \\ \overline{w_j^{\pm}(z)} w_{-j}^{*\pm}(\zeta) &= \left(\frac{r}{\rho}\right)^{\sigma_j} \frac{\overline{c(t)}}{aj} e^{ij(t-\theta)} + \left(\frac{r}{\rho}\right)^{\overline{\sigma_j}} \frac{\overline{c(\theta)}}{aj} e^{-ij(t-\theta)} + O(j^{-2}) \end{aligned} \tag{5.6}$$

Proof. It follows from Theorem 4.1 that for large $|j|$, the corresponding spectral values are in $\mathbb{C}\backslash\mathbb{R}$ (for $b \neq 0$). We have, $\sigma_j^- = \sigma_j^+ = \sigma_j$ and σ_j is given by (4.2). Furthermore, it follows from section 2 that

$$w_j^+(r,t) = r^{\sigma_j}\phi_j(t) + \overline{r^{\sigma_j}\psi_j(t)} \quad \text{and} \quad w_j^-(r,t) = i(r^{\sigma_j}\phi_j(t) - \overline{r^{\sigma_j}\psi_j(t)})$$

with $\phi_j(0) = 1$. For the basic solutions of the adjoint operator, we have (from Theorem 2.2) that

$$\begin{aligned} w_{-j}^{*+}(\rho,\theta) &= i(\rho^{-\sigma_j}X_{-j}(\theta) - \overline{\rho^{-\sigma_j}Z_{-j}(\theta)}) \quad \text{and} \\ w_{-j}^{*-}(\rho,\theta) &= \rho^{-\sigma_j}X_{-j}(\theta) + \overline{\rho^{-\sigma_j}Z_{-j}(\theta)} \end{aligned}$$

with $X_{-j}(0) = 1$. Hence

$$\begin{aligned} w_j^{\pm}(z)w_{-j}^{*\pm}(\zeta) &= 2i\left(\left(\frac{r}{\rho}\right)^{\sigma_j}\phi_j(t)X_{-j}(\theta) - \overline{\left(\frac{r}{\rho}\right)^{\sigma_j}\psi_j(t)Z_{-j}(\theta)}\right) \\ \overline{w_j^{\pm}(z)}w_{-j}^{*\pm}(\zeta) &= 2i\left(\left(\frac{r}{\rho}\right)^{\sigma_j}\psi_j(t)X_{-j}(\theta) - \overline{\left(\frac{r}{\rho}\right)^{\sigma_j}\phi_j(t)Z_{-j}(\theta)}\right) \end{aligned} \tag{5.7}$$

Now, the asymptotic expansions (4.3) and (4.4) give the following products

$$\begin{aligned} \phi_j(t)X_{-j}(\theta) &= \mathrm{e}^{ij(t-\theta)}\left(1 + i\frac{K(t,\theta)}{j} + O(j^{-2})\right) \\ \psi_j(t)X_{-j}(\theta) &= -i\frac{\overline{c(t)}}{2aj}\mathrm{e}^{ij(t-\theta)} + O(j^{-2}) \\ \phi_j(t)Z_{-j}(\theta) &= -i\frac{c(\theta)}{2aj}\mathrm{e}^{ij(t-\theta)} + O(j^{-2}) \\ \psi_j(t)Z_{-j}(\theta) &= O(j^{-2}) \end{aligned} \tag{5.8}$$

Estimates (5.6) of the lemma follow from (5.7) and (5.8) □

Lemma 5.2 *For $j \in \mathbb{Z}^+$ large and σ_j as in* (4.2)*, consider the function*

$$f_j(t) = t^{\sigma_j} - t^{\lambda(j+\nu)}, \qquad 0 < t < 1\,.$$

Then there are $J_0 > 0$ and $C > 0$ such that

$$|f_j(t)| \leq \frac{C}{j^2}, \qquad \forall t \in (0,\ 1),\ j \geq J_0$$

Proof. By using the asymptotic expansion for σ_j given in (4.2) and $\lambda = a+ib$ $(a > 0)$, we write

$$\sigma_j = \left[a(j+\nu) + \frac{\alpha}{j}\right] + i\left[b(j+\nu) + \frac{2\beta}{j^2}\right]$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$, depend on j, but are bounded. Hence,

$$f_j(t) = t^{a(j+\nu)+(\alpha/j)} t^{i[b(j+\nu)+2(\beta/j^2)]} - t^{a(j+\nu)} t^{ib(j+\nu)}$$

We decompose f_j as $f_j = g_j + h_j$ with

$$\begin{aligned} g_j(t) &= t^{a(j+\nu)} t^{i(b(j+\nu)+2(\beta/j^2))} \left(t^{\alpha/j} - 1\right) \\ h_j(t) &= t^{a(j+\nu)} t^{ib(j+\nu)} \left(t^{2i\beta/j^2} - 1\right) \end{aligned}$$

It is enough to verify that both $|g|$ and $|h|$ are $O(1/j^2)$. Since $a > 0$ and j large, then we can assume that g and h are defined at 0 and that $g(0) = h(0) = g(1) = h(1) = 0$.

For the function g, we have $|g(t)| = t^{a(j+\nu)} - t^{a(j+\nu)+(\alpha/j)}$. The maximum of $|g|$ occurs at the point

$$t_* = \left(\frac{a(j+\nu)}{a(j+\nu)+(\alpha/j)}\right)^{j/\alpha}$$

and

$$|g(t_*)| = t_*^{a(j+\nu)} \left(1 - \frac{a(j+\nu)}{a(j+\nu)+(\alpha/j)}\right) \leq \frac{\alpha}{j(a(j+\nu)+(\alpha/j))} \leq \frac{A_1}{j^2}$$

for some positive constant A_1.

For the function h, note that if $\beta = 0$, then $h = 0$. So assume that $\beta \neq 0$. We have

$$|h(t)|^2 = t^{2a(j+\nu)} \left| t^{i\beta/j^2} - t^{-i\beta/j^2} \right|^2 = 4t^{2a(j+\nu)} \sin^2\left(\frac{\beta \ln t}{j^2}\right)$$

For $0 < t \leq 1/2$, we get

$$|h(t)| \leq 2t^{a(j+\nu)} \leq \frac{2}{2^{a(j+\nu)}} \leq \frac{A_2}{j^2}$$

for some $A_2 > 0$. For $1 \geq t \geq (1/2)$, we have

$$\frac{d}{dt}(|h(t)|^2) = 8t^{a(j+\nu)-1} \sin(\frac{\beta \ln t}{j^2}) \left[a(j+\nu) \sin(\frac{\beta \ln t}{j^2}) + \frac{\beta}{j^2} \cos(\frac{\beta \ln t}{j^2}) \right].$$

For j sufficiently large, the critical points of $|h|^2$ in the interval $(1/2\,,\ 1)$ are the solutions of the equation

$$\tan\left(\frac{\beta \ln t}{j^2}\right) = -\frac{\beta}{aj^2(j+\nu)}.$$

Hence,

$$\frac{\beta \ln t}{j^2} = -\arctan(\frac{\beta}{aj^2(j+\nu)}) + k\pi, \qquad k \in \mathbb{Z}.$$

However, since $1 \geq t \geq 1/2$ and j is large, the only possible value of the integer k is $k = 0$. Hence, $|h|$ has a single critical point in $(1/2,\ 1)$:

$$t_* = \exp\left(-\frac{j^2}{\beta}\arctan\left(\frac{\beta}{aj^2(j+\nu)}\right)\right).$$

The function $|h|$ has a maximum at t_* and

$$|h(t_*)|^2 = 4t_*^{2a(j+\nu)}\sin^2(\arctan(\frac{\beta}{aj^2(j+\nu)})) \leq 4\arctan^2(\frac{\beta}{aj^2(j+\nu)}) \leq \frac{A_3}{j^6}$$

for some $A_3 > 0$ $\square$

5.2 Proof of Theorem 5.1

We use the series expansions

$$\begin{aligned}
\frac{\zeta}{\zeta - z} &= \begin{cases} \sum_{j\geq 0}(r/\rho)^{\lambda j}\mathrm{e}^{ij(t-\theta)} & \text{if } r < \rho \\ -\sum_{j\geq 1}(r/\rho)^{-\lambda j}\mathrm{e}^{-ij(t-\theta)} & \text{if } r < \rho \end{cases} \\
L(z,\zeta) &= \begin{cases} \sum_{j\geq 1}(r/\rho)^{\lambda j}\frac{\mathrm{e}^{ij(t-\theta)}}{j} & \text{if } r < \rho \\ \sum_{j\geq 1}(r/\rho)^{-\lambda j}\frac{\mathrm{e}^{-ij(t-\theta)}}{j} & \text{if } r < \rho \end{cases}
\end{aligned}$$

together with (5.2) and (5.6) to decompose $C_1(z,\zeta)$ as follows. For a large integer J_0 and $r < \rho$,

$$C_1 = P_1 + i\sum_{j\geq J_0}\left[\left(\frac{r}{\rho}\right)^{\sigma_j} - \left(\frac{r}{\rho}\right)^{\lambda(j+\nu)}\right]\mathrm{e}^{ij(t-\theta)}\left(1 + i\frac{K}{j} + O(j^{-2})\right) \quad (5.9)$$

where $P_1(z,\zeta)$ consists of the finite collection of terms in the series with index $j < J_0$. Thus $P \in C^1(\Delta_1)$. The second term $(\sum_{j\geq J_0}\cdots)$ on the right of (5.9) is also in $C^1(\Delta_1)$ since

$$\left|\left(\frac{r}{\rho}\right)^{\sigma_j} - \left(\frac{r}{\rho}\right)^{\lambda(j+\nu)}\right| = O(j^{-2})$$

by Lemma 5.2. When $r > \rho$, the decomposition of C_1 takes the form

$$C_1 = \widetilde{P}_1 + i\sum_{j\geq J_0}\left[\left(\frac{r}{\rho}\right)^{-\lambda(j-\nu)} - \left(\frac{r}{\rho}\right)^{\sigma_{-j}}\right]\mathrm{e}^{-ij(t-\theta)}\left(1 - i\frac{K}{j} + O(1/j^2)\right) \quad (5.10)$$

As before, the finite sum $\widetilde{P}_1(z,\zeta) \in C^1(\Delta_2)$. Since $\sigma_{-j} = \lambda(-j+\nu)-(\gamma/j)+O(j^{-2})$ and $r > \rho$, we have

$$\left|\left(\frac{r}{\rho}\right)^{-\lambda(j-\nu)} - \left(\frac{r}{\rho}\right)^{\sigma_{-j}}\right| = \left|\left(\frac{\rho}{r}\right)^{\lambda(j-\nu)} - \left(\frac{\rho}{r}\right)^{-\sigma_{-j}}\right| = O(j^{-2}).$$

Again, the infinite sum on the right of (5.10) is in $C^1(\Delta_2)$. This proves the theorem for the function C_1. Similar arguments can be used for the function C_2 $\square$

5.3 Modified kernels

The following modifications to the kernels Ω_1 and Ω_2 will be used to establish a similarity principle in section 8. For $j_0 \in \mathbb{Z}$, we define $\Omega^{\pm}_{j_0,1}(z,\zeta)$ and $\Omega^{\pm}_{j_0,2}(z,\zeta)$ by

$$\Omega^{\pm}_{j_0,1}(z,\zeta) = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{\mathrm{Re}(\sigma^{\pm}_j) \geq \mathrm{Re}(\sigma^{\pm}_{j_0})} w^{\pm}_j(z) w^{*\pm}_{-j}(\zeta) & \text{if } r < \rho \\ -\dfrac{1}{2} \displaystyle\sum_{\mathrm{Re}(\sigma^{\pm}_j) < \mathrm{Re}(\sigma^{\pm}_{j_0})} w^{\pm}_j(z) w^{*\pm}_{-j}(\zeta) & \text{if } r > \rho \end{cases} \tag{5.11}$$

and

$$\Omega^{\pm}_{j_0,2}(z,\zeta) = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{\mathrm{Re}(\sigma^{\pm}_j) \geq \mathrm{Re}(\sigma^{\pm}_{j_0})} \overline{w^{\pm}_j(z)} w^{*\pm}_{-j}(\zeta) & \text{if } r < \rho \\ -\dfrac{1}{2} \displaystyle\sum_{\mathrm{Re}(\sigma^{\pm}_j) < \mathrm{Re}(\sigma^{\pm}_{j_0})} \overline{w^{\pm}_j(z)} w^{*\pm}_{-j}(\zeta) & \text{if } r > \rho \end{cases} \tag{5.12}$$

Theorem 5.2 *The functions $C_1(z,\zeta)$ and $C_2(z,\zeta)$ given by*

$$\Omega^{\pm}_{j_0,1}(z,\zeta) = i\left(\frac{r}{\rho}\right)^{\sigma^{\pm}_{j_0}} \mathrm{e}^{ij_0(t-\theta)} \left[\frac{\zeta}{\zeta - z} + iK(t,\theta)L(z,\zeta) + C_1(z,\zeta)\right] \tag{5.13}$$

and

$$\begin{aligned} \Omega^{\pm}_{j_0,2}(z,\zeta) &= \frac{\overline{c(t)}}{2a}\left(\frac{r}{\rho}\right)^{\sigma^{\pm}_{j_0}} L(z,\zeta) + \overline{\frac{c(\theta)}{2a}\left(\frac{r}{\rho}\right)^{\sigma^{\pm}_{j_0}} L(z,\zeta)} \\ &\quad + \left(\frac{r}{\rho}\right)^{\sigma^{\pm}_{j_0}} C_2(z,\zeta) \end{aligned} \tag{5.14}$$

have the following properties:

1. $C_{1,2} \in C^1(\mathrm{Int}(\Delta_1) \cup \mathrm{Int}(\Delta_2))$,

2. *for a given* $z = r^\lambda e^{it}$ *with* $0 < r < R$, *the functions* $C_{1,2}(z, \cdot)$ *are in* $L^p(\{(\rho,\theta);\ \rho < R\})$, *for every* $p > 0$, *and*

3. *the functions* $|z-\zeta|^\epsilon C_{1,2}$ *are bounded in the region* $r < R$ *and* $\rho < R$, *for any* $\epsilon > 0$.

Proof. The proof follows similar arguments as those used in the proof of Theorem 5.1. We describe briefly how the properties of C_1 can be established in the region $r < \rho$. We write

$$\Omega^{\pm}_{j_0,1} - i\left(\frac{r}{\rho}\right)^{\sigma^{\pm}_{j_0}} e^{ij_0(t-\theta)}\left[\frac{\zeta}{\zeta - z} + iK(t,\theta)L(z,\zeta)\right] = P_1(z,\zeta) + \sum_{j\geq J_0}$$

where P_1 is the finite sum consisting of terms in the series of $\Omega^{\pm}_{j_0,1}$ with indices $j \leq J_0$ and $\mathrm{Re}(\sigma^{\pm}_j) \geq \mathrm{Re}(\sigma^{\pm}_{j_0})$, and the terms with indices $j \leq J_0$ in the series expansions of $\zeta/(\zeta - z)$ and $L(z,\zeta)$. The infinite sum $\sum\limits_{j\geq J_0}$ can be written as

$$\sum_{j\geq J_0} = i\left(\frac{r}{\rho}\right)^{\sigma^{\pm}_{j_0}} \sum_{j\geq J_0} A_j(z,\zeta)$$

where

$$A_j(z,\zeta) = \left[\left(\frac{r}{\rho}\right)^{\sigma_j - \sigma^{\pm}_{j_0}} - \left(\frac{r}{\rho}\right)^{\lambda(j-j_0)}\right] e^{ij(t-\theta)}\left(1 + i\frac{K(t,\theta)}{j} + O(j^{-2})\right)$$

Since σ_j satisfies the asymptotic expansion (4.2) and since $r < \rho$, arguments similar to those used in the proof of Lemma 5.2 show that

$$\left|\left(\frac{r}{\rho}\right)^{\sigma_j - \sigma^{\pm}_{j_0}} - \left(\frac{r}{\rho}\right)^{\lambda(j-j_0)}\right| = O\left(\frac{1}{j}\right).$$

Thus $\sum\limits_{j\geq J_0}$ has the desired properties of the theorem in Δ_1. Analogous arguments can be used in the region $r > \rho$ and also for the function C_2 □

6 The homogeneous equation $\mathcal{L}u = 0$

In this section, we use the kernels defined in section 5 to obtain series and integral representations of the solutions of the equation $\mathcal{L}u = 0$. Versions of the Laurent series expansion, in terms of the basic solutions, and the Cauchy integral formula are derived. Some consequences of these representations are given.

6.1 Representation of solutions in a cylinder

For $R, \delta \in \mathbb{R}^+$ with $\delta < R$, consider the cylinder $A(\delta, R) = (\delta,\ R) \times \mathbb{S}^1$. Again, let $z = r^\lambda e^{it}$, $\zeta = \rho^\lambda e^{i\theta}$ and Ω_1, Ω_2 denote the functions defined in section 5. We have the following theorem.

Theorem 6.1 *Let $u \in C^0(\overline{A(\delta, R)})$ be a solution of $\mathcal{L}u = 0$. Then*

$$u(z) = \frac{-1}{2\pi} \int_{\partial A(\delta,R)} \frac{\Omega_1(z,\zeta)}{\zeta} u(\zeta) d\zeta + \frac{\overline{\Omega_2(z,\zeta)}}{\overline{\zeta}}\, \overline{u(\zeta)}\, \overline{d\zeta} \tag{6.1}$$

Proof. Let

$$L = \lambda \frac{\partial}{\partial t} - ir\frac{\partial}{\partial r} \quad \text{and} \quad L^* = \lambda \frac{\partial}{\partial \theta} - i\rho \frac{\partial}{\partial \rho},$$

so that $\mathcal{L}u = Lu + i\lambda\nu u - c(t)\overline{u}$ and $-\mathcal{L}^*v = L^*v - i\lambda\nu v + \overline{c(\theta)}\overline{v}$. It follows from the definitions of the kernels $\Omega_{1,2}$ given in (5.2) and (5.3) and from the fact that $\mathcal{L}w_j^\pm = 0$ and $\mathcal{L}^* w_k^{*\pm} = 0$ that the kernels satisfy the following relations

$$\begin{aligned} L^*\Omega_1(z,\zeta) &= i\lambda\nu\Omega_1(z,\zeta) - \overline{c(\theta)}\,\overline{\Omega_2(z,\zeta)} \\ L^*\Omega_2(z,\zeta) &= i\lambda\nu\Omega_2(z,\zeta) - \overline{c(\theta)}\,\overline{\Omega_1(z,\zeta)} \end{aligned} \tag{6.2}$$

Consider the functions $P_1 = \Omega_1 + \Omega_2$ and $P_2 = -i\Omega_1 + i\Omega_2$. Then (6.2) implies that

$$\mathcal{L}^* P_1 = \mathcal{L}^* P_2 = 0\,. \tag{6.3}$$

Let $(r_0, t_0) \in A(\delta, R)$ and $z_0 = r_0^\lambda e^{it_0}$. For $\epsilon > 0$, let

$$D_\epsilon = \{(\rho, \theta) \in \mathbb{R}^+ \times \mathbb{S}^1, \quad |\zeta - z_0| < \epsilon\}\,.$$

Hence, for ϵ small, D_ϵ is diffeomorphic to the disc and is contained in $A(\delta, R)$. We apply Green's identity (1.8) in the region $A(\delta, R)\backslash D_\epsilon$ to each pair $u(\zeta)$, $P_k(z_0, \zeta)$, with $k = 1, 2$. Since, $\mathcal{L}u = 0$ and $\mathcal{L}^*P_k = 0$, then

$$\begin{aligned} &\mathrm{Re}\left[\int_{\partial A} P_k(z_0,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{P_k(z_0,\zeta)}\,\overline{u(\zeta)}\,\frac{\overline{d\zeta}}{\overline{\zeta}}\right] = \\ &\qquad\qquad \mathrm{Re}\left[\int_{\partial D_\epsilon} P_k(z_0,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{P_k(z_0,\zeta)}\,\overline{u(\zeta)}\,\frac{\overline{d\zeta}}{\overline{\zeta}}\right] \end{aligned}$$

Then, after multiplying by i the above identity with $k = 2$ and adding it to the identity with $k = 1$, we obtain

$$\int_{\partial A} (P_1 + iP_2)u\frac{d\zeta}{\zeta} + (\overline{P_1} + i\overline{P_2})\overline{u}\frac{\overline{d\zeta}}{\overline{\zeta}} = \int_{\partial D_\epsilon} (P_1 + iP_2)u\frac{d\zeta}{\zeta} + (\overline{P_1} + i\overline{P_2})\overline{u}\frac{\overline{d\zeta}}{\overline{\zeta}}$$

Since $2\Omega_1 = P_1 + iP_2$ and $2\Omega_2 = P_1 - iP_2$, we get

$$\begin{aligned}\int_{\partial A} \Omega_1(z_0,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{\Omega_2(z_0,\zeta)}\,\overline{u(\zeta)}\,\frac{d\overline{\zeta}}{\overline{\zeta}} = \\ \int_{\partial D_\epsilon} \Omega_1(z_0,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{\Omega_2(z_0,\zeta)}\,\overline{u(\zeta)}\,\frac{d\overline{\zeta}}{\overline{\zeta}}\end{aligned} \tag{6.4}$$

Now, we let $\epsilon \to 0$ in the right side of (6.4). From the estimates (5.5) of the kernels, it follows that the only term that provides a nonzero contribution (as $\epsilon \to 0$) is the term containing $\zeta/(\zeta - z)$ since C_1, C_2 are bounded and $L(z_0,\zeta)$ has a logarithmic growth. That is,

$$\lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \Omega_1 u\frac{d\zeta}{\zeta} + \overline{\Omega_2}\,\overline{u}\,\frac{d\overline{\zeta}}{\overline{\zeta}} = \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} i(r_0/\rho)^{\lambda\nu}\frac{u(\zeta)}{\zeta - z_0}d\zeta = -2\pi u(z_0)$$

This proves the Theorem $\Box$

Theorem 6.2 *Suppose that u is a solution of $\mathcal{L}u = 0$ in the cylinder $A(\delta, R)$ with $0 \le \delta \le R \le \infty$. Then, u has the Laurent series expansion*

$$u(r,t) = \sum_{j\in\mathbb{Z}} a_j^{\pm} w_j^{\pm}(r,t) \tag{6.5}$$

where $a_j^{\pm} \in \mathbb{R}$ are given by

$$a_j^{\pm} = \frac{-1}{2\pi}\mathrm{Re}\int_0^{2\pi} w_{-j}^{*\pm}(R_0,\theta)u(R_0,\theta)id\theta \tag{6.6}$$

where R_0 is any point in $(\delta,\ R)$. Furthermore, there exists $C > 0$ such that

$$|a_j^{\pm}| \le \frac{C}{R_0^{\mathrm{Re}(\sigma_j^{\pm})}}\max_\theta |u(R_0,\theta)| \qquad \forall j \in \mathbb{Z} \tag{6.7}$$

Proof. Let $R_0 \in (\delta,\ R)$, and δ_1, R_1 be such that $\delta < \delta_1 < R_0 < R_1 < R$. For $r \in (\delta_1,\ R_1)$, we apply the integral representation (6.1) in the cylinder $A(\delta_1, R_1)$ to get

$$-2\pi u(r,t) = I_1 - I_2 \tag{6.8}$$

where

$$\begin{aligned} I_1 &= \int_{\rho=R_1} \Omega_1(z,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{\Omega_2(z,\zeta)}\,\overline{u(\zeta)}\,\frac{d\overline{\zeta}}{\overline{\zeta}} \\ I_2 &= \int_{\rho=\delta_1} \Omega_1(z,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{\Omega_2(z,\zeta)}\,\overline{u(\zeta)}\,\frac{d\overline{\zeta}}{\overline{\zeta}}\end{aligned}$$

The series (5.2) and (5.3) for Ω_1 and Ω_2 give

$$I_1 = \sum_{\mathrm{Re}(\sigma_j^{\pm})\ge 0} w_j^{\pm}(r,t)\mathrm{Re}\int_{\rho=R_1} w_{-j}^{*\pm}(\zeta)u(\zeta)\frac{d\zeta}{\zeta}\,.$$

Since $\mathcal{L}u = 0$ and $\mathcal{L}^* w_j^{*\pm} = 0$, then Green's identity gives

$$\mathrm{Re} \int_{\rho=R_1} w_{-j}^{*\pm}(\zeta) u(\zeta) \frac{d\zeta}{\zeta} = \mathrm{Re} \int_0^{2\pi} w_{-j}^{*\pm}(R_0, \theta) u(R_0, \theta) i d\theta \,.$$

Hence,

$$I_1 = -2\pi \sum_{\mathrm{Re}(\sigma_j^\pm) \geq 0} a_j^\pm w_j^\pm(r, t)$$

where $a_j^\pm$ is given by (6.6). A similar calculation shows that

$$I_2 = 2\pi \sum_{\mathrm{Re}(\sigma_j^\pm) < 0} a_j^\pm w_j^\pm(r, t) \,.$$

To estimate the coefficients $a_j^\pm$, recall that

$$w_{-j}^{*\pm}(R_0, \theta) = R_0^{-\sigma_j^\pm} X_{-j}^\pm(\theta) + \overline{R_0^{-\sigma_j^\pm} Z_{-j}^\pm(\theta)} \,.$$

Thus

$$|w_{-j}^{*\pm}(R_0, \theta)| \leq \frac{1}{R_0^{\mathrm{Re}(\sigma_j^\pm)}} \left(|X_{-j}^\pm(\theta)| + |Z_{-j}^\pm(\theta)| \right) \leq \frac{C}{R_0^{\mathrm{Re}(\sigma_j^\pm)}}$$

where $C = \sup\limits_{k,\theta} \left(|X_k^\pm(\theta)| + |Z_k^\pm(\theta)| \right)$. This gives estimate (6.7) $\square$

The following theorem is a direct consequence of Theorem 6.2.

Theorem 6.3 *Let u be a bounded solution of $\mathcal{L}u = 0$ in the cylinder $A(0, R)$. Then u has the series expansion*

$$u(r, t) = \sum_{\mathrm{Re}(\sigma_j^\pm) \geq 0} a_j^\pm w_j^\pm(r, t) \tag{6.9}$$

where $a_j^\pm$ are given by (6.6). If, in addition, u is continuous on $\overline{A(0, R)}$, then the above summation is taken over the spectral values $\sigma_j^\pm$ satisfying $\mathrm{Re}(\sigma_j^\pm) > 0$ *or* $\sigma_j^\pm = 0$.

6.2 Cauchy integral formula

For a subset $U \subset \mathbb{R} \times \mathbb{S}^1$, we set $\partial_0 U = \partial U \backslash S_0$, where $S_0 = \{0\} \times \mathbb{S}^1$. We have the following integral representation that generalizes the classical Cauchy integral formula.

Theorem 6.4 *Let U be an open and bounded subset of $\mathbb{R}^+ \times \mathbb{S}^1$ such that ∂U consists of finitely many simple closed and piecewise smooth curves. Let $u \in C^0(\overline{U}\backslash S_0)$ be such that $\mathcal{L}u = 0$. Then, for $(r,t) \in U$, we have*

$$u(r,t) = \frac{-1}{2\pi}\int_{\partial_0 U} \Omega_1(z,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{\Omega_2(z,\zeta)}\,\overline{u(\zeta)}\,\frac{d\overline{\zeta}}{\overline{\zeta}} \tag{6.10}$$

Proof. For $\delta > 0$, define $U_\delta = U\backslash A(0,\delta)$. Let $(r_0,t_0) \in U$. Choose $\epsilon > 0$ and $\delta > 0$ small enough so that $(r_0,t_0) \in U_\delta$ and $D_\epsilon \subset U_\delta$, where $D_\epsilon = \{(\rho,\theta);\ |\zeta - z_0| < \epsilon\}$. Arguments similar to those used in the proof of Theorem 6.1 show that

$$u(r_0,t_0) = \frac{-1}{2\pi}\int_{\partial U_\delta} \Omega_1(z,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{\Omega_2(z,\zeta)}\,\overline{u(\zeta)}\,\frac{d\overline{\zeta}}{\overline{\zeta}}\,. \tag{6.11}$$

If $S_0 \cap \partial U = \emptyset$, then for δ small enough $U_\delta = U$ and the theorem is proved in this case. If $S_0 \cap \partial U \neq \emptyset$, let $\Gamma_\delta = \partial U_\delta \cap \big(\{\delta\} \times \mathbb{S}^1\big)$. That is, Γ_δ is the part of ∂U_δ contained in the circle $r = \delta$. Going back to the definition of the kernels, we obtain

$$\int_{\Gamma_\delta} \Omega_1(z,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{\Omega_2(z,\zeta)}\,\overline{u(\zeta)}\,\frac{d\overline{\zeta}}{\overline{\zeta}} = -\sum_{\mathrm{Re}(\sigma_j^\pm)<0} w_j^\pm(r_0,t_0)\mathrm{Re}\int_{\Gamma_\delta} w_{-j}^{*\pm}(\zeta)u(\zeta)\frac{d\zeta}{\zeta} \tag{6.12}$$

Since there exists $C > 0$ such that

$$|w_{-j}^{*\pm}(\rho,\theta)| \le C\rho^{-\mathrm{Re}(\sigma_j^\pm)} \qquad \forall j \in \mathbb{Z}$$

then, it follows from (6.12), that

$$\left|\int_{\Gamma_\delta} \Omega_1(z,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{\Omega_2(z,\zeta)}\,\overline{u(\zeta)}\,\frac{d\overline{\zeta}}{\overline{\zeta}}\right| \le 2\pi C||u||_0 \sum_{\mathrm{Re}(\sigma_j^\pm)<0} \rho^{-\mathrm{Re}(\sigma_j^\pm)}|w_j^\pm(r_0,t_0)|$$

Therefore,

$$\lim_{\delta\to 0}\int_{\Gamma_\delta} \Omega_1(z,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{\Omega_2(z,\zeta)}\,\overline{u(\zeta)}\,\frac{d\overline{\zeta}}{\overline{\zeta}} = 0$$

and (6.10) follows from (6.11) when we let $\delta \to 0$ $\square$

The following theorem extends the Cauchy integral formula to include the points on the characteristic circle S_0, when the solution is continuous up to the boundary.

Theorem 6.5 *Suppose that $\mathcal{L}$ has no spectral values in $i\mathbb{R}^*$. Let U be an open, bounded subset of $\mathbb{R}^+ \times \mathbb{S}^1$, such that ∂U consists of finitely many*

simple closed and piecewise smooth curves and with $S_0 \subset \partial U$. Let $u \in C^0(\overline{U})$ be such that $\mathcal{L}u = 0$. Then, for $(r,t) \in U \cup S_0$, we have

$$u(r,t) = \frac{-1}{2\pi} \int_{\partial_0 U} \Omega_1(z,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{\Omega_2(z,\zeta)}\,\overline{u(\zeta)}\,\frac{d\overline{\zeta}}{\overline{\zeta}} \tag{6.13}$$

Proof. We know from Theorem 6.4 that (6.13) holds for $r > 0$. We need to verify that it holds at the points $(0,t) \in S_0$. Since $\mathcal{L}$ has no purely imaginary spectral values, then Ω_1 and Ω_2 are well defined on S_0. We have $\Omega_1(0,t) = \Omega_2(0,t) = 0$ if 0 is not a spectral value of $\mathcal{L}$ and if 0 is a spectral value, with say multiplicity 2, then

$$\begin{aligned} \Omega_1(0,t,\rho,\theta) &= \underline{f^+_{j_0}(t)}g^+_{-j_0}(\theta) + \underline{f^-_{j_0}(t)}g^-_{-j_0}(\theta) \\ \Omega_2(0,t,\rho,\theta) &= \overline{f^+_{j_0}(t)}g^+_{-j_0}(\theta) + \overline{f^-_{j_0}(t)}g^-_{-j_0}(\theta) \end{aligned}$$

where $f^{\pm}_{j_0}(t)$ are the basic solutions of $\mathcal{L}$ with exponent 0 and $g^{\pm}_{-j_0}(t)$ are the basic solutions of $\mathcal{L}^*$ with exponent 0.

When 0 is not a spectral value, (6.13) holds for $r = 0$ by letting $r \to 0$ in (6.10). In this case $u \equiv 0$ on S_0. When 0 is spectral value, then since $S_0 \subset \partial U$, $u(r,t)$ has a Laurent series expansion in a cylinder $A(0,\delta) \subset U$. In particular,

$$\begin{aligned} u(0,t) &= a^+ f^+_{j_0}(t) + a^- f^-_{j_0}(t) \\ &= \frac{-1}{2\pi} \int_{\partial_0 U} \Omega_1(0,t,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{\Omega_2(0,t,\zeta)}\,\overline{u(\zeta)}\,\frac{d\overline{\zeta}}{\overline{\zeta}} \quad \square \end{aligned}$$

Remark 6.1 It follows from this Theorem that if $\mathcal{L}$ has no spectral values on $i\mathbb{R}^*$, then all bounded solutions of $\mathcal{L}u = 0$ in a cylinder $A(0,R)$ are continuous up to S_0. If $\mathcal{L}$ has spectral values on $i\mathbb{R}^*$, then there are bounded solutions on $A(0,R)$ that are not continuous up to S_0. In fact, the basic solution $r^{i\tau}\phi(t) + r^{-i\tau}\psi(t)$ is such a solution when $i\tau \in \mathrm{Spec}(\mathcal{L})$. Note also that the number of spectral values on $i\mathbb{R}^*$ is at most finite (this follows from the asymptotic expansion of σ_j).

6.3 Consequences

We give here some consequences of the above representations theorems. First we define the order of a solution along S_0. We say that a solution, u, of $\mathcal{L}u = 0$ in a cylinder $A(0,R)$ has a zero or a pole of order $s = \mathrm{Re}(\sigma_{j_0})$ (with $\sigma_{j_0} = \sigma^+_{j_0}$ or $\sigma^-_{j_0}$) along the circle S_0 if, in the Laurent series expansion of u, all the coefficients $a^{\pm}_j$ corresponding to $\mathrm{Re}(\sigma^{\pm}_j) < s$ are zero. That is

$$u(r,t) = \sum_{\mathrm{Re}(\sigma^{\pm}_j) \geq s} a^{\pm}_j w^{\pm}_j(r,t)\,.$$

We have the following uniqueness result.

Theorem 6.6 *Suppose that the spectral values of $\mathcal{L}$ satisfy the following condition*

$$\mathrm{Re}(\sigma_j) = \mathrm{Re}(\sigma_k) \implies \sigma_j = \sigma_k, \qquad \forall \sigma_j, \sigma_k \in \mathrm{Spec}(\mathcal{L})\,. \tag{6.14}$$

Let u be a solution of $\mathcal{L}u = 0$ in a cylinder $A(0,R)$. Suppose that u is of finite order along S_0 and that there is a sequence of points $(r_k, t_k) \in A(0,R)$ such that $r_k \to 0$ and $u(r_k, t_k) = 0$ for every $k \in Z^+$. Then $u \equiv 0$

Proof. By contradiction, suppose that $u \not\equiv 0$. Let s be the order of u on S_0. First, consider the case where $s \in \mathbb{R}$ is a spectral value (say of multiplicity 2). Let $r^s f^+(t)$ and $r^s f^-(t)$ be the corresponding basic solutions. The function u has the form

$$u(r,t) = r^s(a^- f^-(t) + a^+ f^+(t)) + o(r^s).$$

The functions f^+ and f^- are independent solutions of the first order differential equation (2.3) and $a^\pm \in \mathbb{R}$ (not both zero). We can assume $t_k \to t_0$ as $k \to \infty$. It follows from the above representation of u and from the hypothesis $u(r_k, t_k) = 0$ that $\lim_{k\to\infty}(u(r_k, t_k)/r_k^s) = 0$. Consequently,

$$a^- f(t_0) + a^+ f^+(t_0) = 0.$$

Thus, the solution $a^- f^- + a^+ f^+$ of (2.3) is identically zero (by uniqueness). Hence, $a^- = a^+ = 0$ which is a contradiction.

If s is not a spectral value, then (by condition (6.14)), it must be the real part of a unique spectral value $\sigma \in \mathbb{C}\backslash\mathbb{R}$. The corresponding $\mathbb{R}$-independent basic solutions are

$$r^{s+i\beta}\phi(t) + \overline{r^{s+i\beta}\psi(t)} \quad \text{and} \quad i(r^{s+i\beta}\phi(t) - \overline{r^{s+i\beta}\psi(t)})$$

with $\beta \in \mathbb{R}^*$ and $|\phi(t)| > |\psi(t)|$ for every $t \in \mathbb{R}$. The Laurent series of u starts as a linear combination of these two basic solution and u can then be written as

$$u(r,t) = r^s\left((a^+ + ia^-)r^{i\beta}\phi(t) + (a^+ - ia^-)\overline{r^{i\beta}\psi(t)}\right) + r^\tau \Phi(r,t)$$

with $a^\pm \in \mathbb{R}$ (not both zero), $\tau > s$ and Φ a bounded function. It follows from the assumption $u(r_k, t_k) = 0$ that for every $k \in \mathbb{Z}$, we have

$$(a^+ + ia^-)\phi(t_k) + (a^+ - ia^-)\overline{r_k^{2i\beta}\psi(t_k)} + r_k^{\tau - s - i\beta}\Phi(r_k, t_k) = 0,$$

But this is only possible when $a^+ + ia^- = 0$ (since $|\phi| > |\psi|$, $r_k \to 0$ and Φ bounded) $\square$

The next theorem deals with sequences of solutions that converge on the distinguished boundary $\partial_0 U = \partial U \backslash S_0$.

Theorem 6.7 *Let U be an open and bounded subset of $\mathbb{R}^+\times\mathbb{S}^1$ whose boundary consists of finitely many simple, closed and piecewise smooth curves. Let $u_n(r,t)$ be a sequence of bounded functions with $u_n \in C^0(\overline{U}\backslash S_0)$ such that $\mathcal{L}u_n = 0$ for every n. If u_n converges uniformly on $\partial_0 U = \partial U\backslash S_0$, then u_n converges uniformly on $\overline{U}\backslash S_0$ to a solution u of $\mathcal{L}u = 0$.*

Proof. For $(r,t) \in \partial_0 U$, let $\Phi(r,t) = \lim_{n\to\infty} u_n(r,t)$. The function $u(r,t)$ defined in U by

$$u(r,t) = \frac{-1}{2\pi}\int_{\partial_0 U} \Omega_1(r,t,\zeta)\Phi(\zeta)\frac{d\zeta}{\zeta} + \overline{\Omega_2(r,t,\zeta)}\,\overline{\Phi(\zeta)}\,\frac{d\overline{\zeta}}{\overline{\zeta}}$$

solves $\mathcal{L}u = 0$ (since for each fixed ζ, $\mathcal{L}\Omega_1(z,\zeta) = \mathcal{L}\,\overline{\Omega_2(z,\zeta)} = 0$). Now, the Cauchy integral formula applied to u_n, shows that u is the uniform limit of u_n inside U $\square$

The following Liouville property is a direct consequence of the Laurent series expansion and estimate (6.7) of the coefficients.

Theorem 6.8 *Let u be a bounded solution of $\mathcal{L}u = 0$ in $\mathbb{R}^+\times\mathbb{S}^1$. Then*

$$u(r,t) = \sum_{\mathrm{Re}(\sigma_j^{\pm})=0} a_j^{\pm} w_j^{\pm}(r,t)\,.$$

In particular, if $\mathcal{L}$ has no spectral values on $i\mathbb{R}$, then $u \equiv 0$.

Another consequence of the Laurent series representation is to patch together solutions from both sides of the characteristic circle S_0. More precisely we have the following theorem.

Theorem 6.9 *Suppose that $\mathcal{L}$ has no spectral values in $i\mathbb{R}^*$. Then we have the following.*

1. *If 0 is not a spectral value, then any bounded solution of $\mathcal{L}u = 0$ in the cylinder $(-R,\ R)\times\mathbb{S}^1$ is continuous on the circle S_0.*

2. *If 0 is a spectral value (say with multiplicity 2), let $g^{\pm}(t)$ be the basic solutions of $\mathcal{L}^*$ with exponent 0. Then a bounded solution u of $\mathcal{L}u = 0$ in $((-R,\ 0)\cup(0,\ R))\times\mathbb{S}^1$ is continuous on $(-R,\ R)\times\mathbb{S}^1$ if and only if*

$$\mathrm{Re}\int_0^{2\pi} g^{\pm}(\theta)u(\delta,\theta)d\theta = \mathrm{Re}\int_0^{2\pi} g^{\pm}(\theta)u(-\delta,\theta)d\theta$$

for some $\delta \in (0,\ R)$.

7 The nonhomogeneous equation $\mathcal{L}u = F$

After we extend the Cauchy integral formula to include the nonhomogeneous case, we define an integral operator for the nonhomogeneous equation $\mathcal{L}u = F$. Throughout the remainder of this paper U will denote an open and bounded set in $\mathbb{R}^+ \times \mathbb{S}^1$ whose boundary consists of finitely many simple, closed and piecewise smooth curves.

7.1 Generalized Cauchy Integral Formula

The following generalization of the Cauchy integral formula will be used later.

Theorem 7.1 *Suppose that $F(r,t)$ is a function in U such that $\dfrac{F}{r} \in L^p(U)$ with $p \geq 1$. If equation $\mathcal{L}u = F$ has a solution $u \in C^0(\overline{U})$, then*

$$\begin{aligned} u(r,t) &= \frac{-1}{2\pi} \int_{\partial_0 U} \Omega_1(r,t,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{\Omega_2(r,t,\zeta)}\,\overline{u(\zeta)}\,\frac{d\overline{\zeta}}{\overline{\zeta}} - \\ &\qquad \frac{1}{2\pi} \iint_U \Big[\Omega_1(r,t,\zeta)F(\zeta) + \overline{\Omega_2(r,t,\zeta)}\,\overline{F(\zeta)}\Big] \,\frac{d\rho d\theta}{\rho} \end{aligned} \tag{7.1}$$

Proof. For $\delta > 0$, let $U_\delta = U \backslash A(0,\delta)$. Let $z_0 \in U$ and choose $\delta > 0$ so that $z_0 \in U_\delta$. Green's identity (1.8) and arguments similar to those used in the proof of the Cauchy integral formula show that

$$\begin{aligned} -2\pi u(z_0) = & \int_{\partial U_\delta} \Omega_1(r,t,\zeta)u(\zeta)\frac{d\zeta}{\zeta} + \overline{\Omega_2(r,t,\zeta)}\,\overline{u(\zeta)}\,\frac{d\overline{\zeta}}{\overline{\zeta}} + \\ &\qquad \iint_{U_\delta} \Big[\Omega_1(r,t,\zeta)F(\zeta) + \overline{\Omega_2(r,t,\zeta)}\,\overline{F(\zeta)}\Big] \,\frac{d\rho d\theta}{\rho} \end{aligned}$$

Since $(F/r) \in L^p$ with $p \geq 1$, then the limits of the above integrals as $\delta \to 0$ give (7.1) $\square$

For the adjoint operator, we have the following.

Theorem 7.2 *Let $v(\rho,\theta) \in C^0(\overline{U})$ be such that $\dfrac{\mathcal{L}^* v}{\rho} \in L^p(U)$ with $p \geq 1$. Then*

$$\begin{aligned} v(\rho,\theta) &= \frac{-1}{2\pi} \int_{\partial_0 U} \Omega_1(z,\rho,\theta)v(z)\frac{dz}{z} + \overline{\Omega_2(z,\rho,\theta)}\,\overline{v(z)}\,\frac{d\overline{z}}{\overline{z}} + \\ &\qquad \frac{1}{2\pi} \iint_U \Big[\Omega_1(z,\rho,\theta)\mathcal{L}^* v(z) + \overline{\Omega_2(z,\rho,\theta)}\,\overline{\mathcal{L}^* v(z)}\Big] \,\frac{dr dt}{r} \end{aligned} \tag{7.2}$$

Proof. Notice that the kernels $\Omega_1(z,\zeta)$ and $\Omega_2(z,\zeta)$ satisfy

$$L\Omega_1 = -i\lambda\nu\Omega_1 + c(t)\Omega_2 \quad \text{and} \quad L\overline{\Omega_2} = -i\lambda\nu\overline{\Omega_2} + c(t)\overline{\Omega_1}$$

where $L = \frac{\partial}{\partial t} - ir\frac{\partial}{\partial r}$. Arguments similar to those used in the proofs of Theorems 6.4 and 7.1 lead to (7.2). The functions P_1 and P_2 used in the proof of Theorem 6.4 need now to be replaced by the functions $Q_1 = \Omega_1 + \overline{\Omega_2}$ and $Q_2 = -i\Omega_1 + i\,\overline{\Omega_2}$ $\square$

7.2 The integral operator T

We define the operator T and the appropriate L^p-spaces in which it acts to produce Hölder continuous solutions. For an open set $U \subset \mathbb{R}^+ \times \mathbb{S}^1$ as before and such that $S_0 \subset \partial U$, we denote by $L^p_a(U)$ the Banach space of functions $F(r,t)$ such that $\frac{F(r,t)}{r^a}$ is integrable in U with the norm

$$||F||_{p,a} = \left(\iint_U \left|\frac{F(r,t)}{r^a}\right|^p r^{2a-1} dr dt\right)^{\frac{1}{p}} .$$

Note that if $\Phi : \mathbb{R}^+ \times \mathbb{S}^1 \longrightarrow \mathbb{C}^*$ is the diffeomorphism induced by the first integral z. That is, $\Phi(r,t) = r^\lambda e^{it}$, then $F \in L^p_a(U)$ means that the push forward $\widetilde{F} = F \circ \Phi^{-1}$ satisfies $\frac{\widetilde{F}(z)}{z} \in L^p(\Phi(U))$.

We define the integral operator T by

$$TF(r,t) = \frac{-1}{2\pi} \iint_U \left[\Omega_1(r,t,\zeta)F(\zeta) + \overline{\Omega_2(r,t,\zeta)}\,\overline{F(\zeta)}\right] \frac{d\rho d\theta}{\rho} \tag{7.3}$$

When $\mathcal{L}$ has no purely imaginary spectral values, i.e. $\mathrm{Spec}(\mathcal{L}) \cap i\mathbb{R}^* = \emptyset$, we have the following theorem.

Theorem 7.3 *Assume* $\mathrm{Spec}(\mathcal{L}) \cap i\mathbb{R}^* = \emptyset$. *Let $U \subset A(0,R)$ be an open set as above. The function TF defined by* (7.3) *satisfies the followings.*

1. *There exist positive constants C and δ, independent on U and R, such that for every $(r,t) \in U$*

$$|TF(r,t)| \leq CR^\delta ||F||_{p,a} \tag{7.4}$$

   for every $F \in L^p_a(U)$ with $p > 2/(1-\nu)$;

2. *the function TF satisfies the equation $\mathcal{L}TF = F$; and*

3. *the function TF is Hölder continuous on $\overline{U}$;*

Furthermore, if 0 *is not a spectral value, then $TF(0,t) \equiv 0$*

Proof. We use the estimates on Ω_1 and Ω_2 of Theorem 5.1 to write

$$-2\pi u(r,t) = I_1 + I_2 + I_3 + I_4\,,$$

where

$$\begin{aligned}
I_1 &= i\iint_U \left(\frac{r}{\rho}\right)^{\lambda\nu} \frac{\zeta}{\zeta - z} F(\zeta)\frac{d\rho d\theta}{\rho} \\
I_2 &= -\iint_U \left(\frac{r}{\rho}\right)^{\lambda\nu} K(t,\theta)L(z,\zeta)F(\zeta)\frac{d\rho d\theta}{\rho} \\
I_3 &= \frac{1}{2a}\iint_U \left[\left(\frac{r}{\rho}\right)^{\overline{\lambda}\nu} c(t)\overline{L(z,\zeta)} + c(\theta)\left(\frac{r}{\rho}\right)^{\lambda\nu} L(z,\zeta)\right]\overline{F(\zeta)}\frac{d\rho d\theta}{\rho} \\
I_4 &= \iint_U \left[C_1(z,\zeta)F(\zeta) + \overline{C_2(z,\zeta)}\,\overline{F(\zeta)}\right]\frac{d\rho d\theta}{\rho}
\end{aligned}$$

We use the substitution $\zeta = \Phi(\rho,\theta) = \rho^\alpha e^{i\theta}$ to estimate I_1. We find

$$|I_1| \le r^{a\nu}\iint_{\Phi(U)} \frac{|\widetilde{F}(\zeta)|}{|\zeta - z||\zeta|^{1+\nu}} d\xi d\eta\,,$$

where we have set $\widetilde{F} = F\circ\Phi^{-1}$ and $\zeta = \xi + i\eta$. Since $\dfrac{\widetilde{F}}{\zeta} \in L^p(\Phi(U))$ with $p > 2/(1-\nu)$, and since $\Phi(U)$ is contained in the disc $D(0,R^a) \subset \mathbb{C}$, then Hölder inequality can be used to show that there are constants C and δ so that $|I_1| \le r^{a\nu}CR^\delta ||\widetilde{F}||_p$. Furthermore these constants are independent on $\Phi(U)$ and R. Because of the logarithmic type growth of $L(z,\zeta)$ and the boundedness of the functions C_1 and C_2, analogous arguments can be used to show that $|I_k| \le CR^\delta ||F||_{p,a}$ for $k = 2,3,4$.

Now we verify that $u = TF$ solves $\mathcal{L}u = F$ in the sense of distributions. Let $\psi \in C^1_0(U)$ be a test function. The generalized Cauchy integral formula (7.2) applied to ψ gives

$$\psi(\rho,\theta) = \frac{1}{2\pi}\iint_U \left[\Omega_1(z,\rho,\theta)\mathcal{L}^*\psi(z) + \overline{\Omega_2(z,\rho,\theta)}\,\overline{\mathcal{L}^*\psi(z)}\right]\frac{drdt}{r} \qquad (7.5)$$

The definition (7.3) of the operator T and estimate (7.4) give

$$\begin{aligned}
2 < TF, \mathcal{L}^*\psi > &= \iint_U \left[TF(z)\mathcal{L}^*\psi(z) + \overline{TF(z)}\,\overline{\mathcal{L}^*\psi(z)}\right]\frac{drdt}{r} \\
&= \iint_U \left[F(\zeta)\psi(\zeta) + \overline{F(\zeta)}\,\overline{\psi(\zeta)}\right]\frac{d\rho d\theta}{\rho} \\
&= 2 < F, \psi >
\end{aligned}$$

This shows that $\mathcal{L}TF = F$.

Next, we prove that TF is Hölder continuous. Since the equation is elliptic away from the circle S_0, it is enough to prove the regularity of TF on S_0. For this, we consider the case when 0 is not a spectral value. Then $i\mathbb{R}\cap\mathrm{Spec}(\mathcal{L}) = \emptyset$ and $\Omega_1(0,t,\zeta) = \Omega_2(0,t,\zeta) = 0$. Hence $TF(0,t) = 0$. Since TF satisfies $\mathcal{L}TF = F$, then its pushforward $V(z) = TF\circ\Phi^{-1}(z)$ via the first integral satisfies the generalized CR equation

$$V_{\overline{z}} = \frac{i\lambda\nu}{2ia\overline{z}}V - \frac{\widetilde{c}(z)}{2ia\overline{z}}\overline{V} - \frac{\widetilde{F}(z)}{2ia\overline{z}}$$

where $\widetilde{c}$ and $\widetilde{F}$ are the pushforwards of c and F. We will use the classical results on the CR equation (see [17]) to show that V is Hölder continuous. We rewrite the above equation as

$$V_{\overline{z}} = \frac{G(z)}{z} \tag{7.6}$$

where

$$G(z) = \frac{i\lambda\nu z}{2ia\overline{z}}\, V(z) - \frac{z\widetilde{c}(z)}{2ia\overline{z}}\,\overline{V(z)} - \frac{z\widetilde{F}(z)}{2ia\overline{z}}\,.$$

Note that since $\widetilde{c}$ and V are bounded functions and since $(\widetilde{F}/z) \in L^p$, then $G \in L^p(\Phi(U))$ with $p > 2$. The solution of (7.6) can then be written as $V(z) = \dfrac{W(z)}{z}$ where W is the solution of the equation $W_{\overline{z}} = G$. We know that W is Hölder continuous and has the form

$$W(z) = H(z) - \frac{1}{\pi}\iint_{\Phi(U)} \frac{G(\zeta)}{\zeta - z} d\xi d\eta$$

where H is a holomorphic function in $\Phi(U)$. Since $V(z) = W(z)/z$ satisfies $V(0) = 0$, then necessarily $W(0) = 0$ and it vanishes to an order > 1 at 0. Thus $|V(z)| \leq K|z|^\tau$ for some positive constants K and τ. This means that TF is Hölder continuous on S_0.

Finally we consider the case when 0 is a spectral value of $\mathcal{L}$ (say, with multiplicity 2). Let $f^\pm_{j_0}(t)$ and $g^\pm_{-j_0}(\theta)$ be the basic solutions of $\mathcal{L}$ and $\mathcal{L}^*$ with exponents 0. We have then

$$\Omega_1(0,t,\zeta) = \frac{1}{2} f^\pm_{j_0}(t) g^\pm_{-j_0}(\theta) \quad \text{and} \quad \Omega_2(0,t,\zeta) = \frac{1}{2}\overline{f^\pm_{j_0}(t)}\, g^\pm_{-j_0}(\theta).$$

The value of TF on S_0 is found to be

$$TF(0,t) = A^+ f^+_{j_0}(t) + A^- f^-_{j_0}(t)\,,$$

where

$$A^\pm = \frac{-1}{2\pi}\mathrm{Re}\iint_U g^\pm_{-j_0}(\theta) F(\zeta) \frac{d\rho d\theta}{\rho}\,.$$

Hence $TF(0,t)$ solves the homogeneous equation $\mathcal{L}u = 0$. Let $v(r,t) = TF(r,t) - TF(0,t)$. The function v satisfies $\mathcal{L}v = F$ and $v(0,t) = 0$. The push forward arguments, used in the case when 0 is not a spectral value, can be used again for the function v to establish that $|v(r,t)| \leq Cr^\tau$ with τ and C positive $\square$

In general, when $\mathcal{L}$ has spectral values on $i\mathbb{R}$, we can define $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ as in (5.2) and (5.3) except that the terms corresponding to $\sigma^\pm_j$ that are in $i\mathbb{R}$ are missing from the sums. That is, if $w^\pm_1, \cdots, w^\pm_p$ denotes the

collection of basic solutions of $\mathcal{L}$ with exponents in $i\mathbb{R}$, and $w_1^{*\pm}, \cdots, w_p^{*\pm}$ the corresponding collection of basic solutions of the adjoint $\mathcal{L}^*$, then

$$\widehat{\Omega}_k(z,\zeta) = \Omega_k(z,\zeta) - \sum_{k=1}^{p} w_k^{\pm}(z) w_k^{*\pm}(\zeta)\,. \tag{7.7}$$

We define the modified operator $\widehat{T}$ by

$$\widehat{T}F(r,t) = \frac{-1}{2\pi}\iint_U \left[\widehat{\Omega}_1(r,t,\zeta)F(\zeta) + \overline{\widehat{\Omega}_2(r,t,\zeta)}\,\overline{F(\zeta)}\right]\frac{d\rho d\theta}{\rho}\,. \tag{7.8}$$

Arguments similar to those used in the proof of Theorem 7.3 establish the following result.

Theorem 7.4 *Let $U \subset A(0,R)$ be as above. Then the function $\widehat{T}F$ defined by* (7.8) *satisfies properties* 1, 2, *and* 3 *of Theorem* 7.3 *and* $\widehat{T}F(0,t) = 0$.

7.3 Compactness of the operator T

Theorem 7.5 *Suppose that $\mathcal{L}$ has no spectral values in $i\mathbb{R}^*$. Then, for $p > 2/(1-\nu)$, the operator $T : L^p_a(U) \longrightarrow C^0(\overline{U})$ is compact.*

Proof. Let $R > 0$ be such that $U \subset A(0,R)$. A function in $L^p_a(U)$ can be considered in $L^p_a(A(0,R))$ by extending as 0 on $A(0,R)\backslash U$. Denote by T_R the operator T on the cylinder $A(0,R)$ and set

$$\widehat{T}_R F(r,t) = T_R F(r,t) - T_R F(0,t)$$

Thus

$$\widehat{T}_R F(r,t) = \frac{-1}{2\pi}\iint_{A(0,R)} \left[\widehat{\Omega}_1(z,\zeta)F(\zeta) + \overline{\widehat{\Omega}_2(z,\zeta)}\,\overline{F(\zeta)}\right]\frac{d\rho d\theta}{\rho}$$

where $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ are defined by (5.2) and (5.3), respectively, except that the terms corresponding to the spectral value $\sigma_j = 0$ are missing. In particular, if 0 is not a spectral value, then $\widehat{\Omega}_1 = \Omega_1$ and $\widehat{\Omega}_2 = \Omega_2$. Note that $\widehat{\Omega}_1(0,t) = 0$ and $\widehat{\Omega}_2(0,t) = 0$. The operator $\widehat{T}_R F$ satisfies the properties of Theorem 7.3 and $\widehat{T}_R F(0,t) = 0$. To show that T is compact, it is enough to show that $\widehat{T}_R$ is compact.

Let $B \subset L^p_a(U)$ be a bounded set. We need to show that $\widehat{T}_R(B)$ is relatively compact in $C^0(\overline{U})$. Let $M > 0$ be such that $||F||_{p,a} \leq M$ for every $F \in B$. It follows from Theorem 7.3 that $\widehat{T}_R(B)$ is bounded (by $CR^\delta M$). Now we show the equicontinuity of $\widehat{T}_R(B)$. First along S_0. For $\epsilon > 0$, let $r_0 > 0$ be such that $Cr_0^\delta M < (\epsilon/2)$. We have then

$$\widehat{T}_R F(z) = \widehat{T}_{r_0} F(z) + \widehat{T}_{A(r_0,R)} F(z)$$

where $\widehat{T}_{A(r_0,R)}$ denotes the integral operator over the cylinder $A(r_0,R)$. Let r_0 be small enough so that

$$E = \max_{r<(r_0/2),\ r_0<\rho<R} (|\widehat{\Omega}_1(r,t,\rho,\theta)| + |\widehat{\Omega}_2(r,t,\rho,\theta)|) \ < \ \frac{\epsilon}{2M(\pi R^{2a})^{1/q}}$$

where q is such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. For $r < r_0/2$, we have then

$$|\widehat{T}_{r_0} F(r,t)| \ \le \ C r_0^{\delta} M \ \le \ \frac{\epsilon}{2}$$

and

$$|\widehat{T}_{A(r_0,R)} F(r,t)| \le \iint_{A(r_0,R)} (|\widehat{\Omega}_1| + |\widehat{\Omega}_2|) \frac{|F(\zeta)|}{\rho} \, d\rho d\theta \le E ||F||_{p,a} \le \frac{\epsilon}{2}$$

This estimate is obtained from Hölder's inequality and the above estimate on E. Thus, $|\widehat{T}_R F(r,t)| \le \epsilon$ and so $\widehat{T}_R B$ is equicontinuous on S_0.

Next, let $(r_1, t_1) \in U$ with $r_1 < r_0/4$. Set $z_1 = r_1^{\lambda} \mathrm{e}^{it_1}$ and $z = r^{\lambda} \mathrm{e}^{it}$. If $|z - z_1| < r_0^a/4$, then $r < r_0/2$ and the above argument gives

$$|\widehat{T}_R F(z) - \widehat{T}_F(z_1)| \le |\widehat{T}_R(z)| + |\widehat{T}_R(z_1)| \le 2\epsilon, \qquad \forall F \in B$$

Finally, suppose that $r_1 > r_0/4$. Let b be such that $0 < b < r_0/4$. We write

$$\widehat{T}_R F(z) - \widehat{T}_R F(z_1) = \widehat{T}_b F(z) - \widehat{T}_b F(z_1) + \widehat{T}_{A(b,R)} F(z) - \widehat{T}_{A(b,R)} F(z_1)$$

After using Hölder's inequality we obtain

$$|\widehat{T}_b F(z) - \widehat{T}_b F(z_1)| \ \le \ C S(b) ||F||_{p,a}$$

where

$$S(b) = \max_P (|\widehat{\Omega}_1(z,\zeta) - \widehat{\Omega}_1(z_1,\zeta)| + |\widehat{\Omega}_2(z,\zeta) - \widehat{\Omega}_2(z_1,\zeta)|)\,,$$

and where the maximum is taken over the set P of points satisfying $\rho < b$, $|r - r_1| < b$, $r > r_0/4$ and $r_1 > r_0/4$. The continuity of the kernels in the region $\rho < b$ and $r > r_0/4$ implies that if b is small enough, then $S(b) < \epsilon/(2MC)$ and consequently

$$|\widehat{T}_b F(z) - \widehat{T}_b F(z_1)| \ \le \ \frac{\epsilon}{2}$$

Finally, for $\widehat{T}_{A(b,R)} F$, it suffices to notice that it solves the equation $\mathcal{L}u = F$ in the cylinder $A(r_0/4, R)$. In this cylinder, the equation is elliptic and the classical theory of generalized analytic function ([17] Chapter 7) implies that the family $\widehat{T}_{A(b,R)} B$ is equicontinuous. $\square$

8 The semilinear equation

In this section, we make use of the operator T and of its modified version, through the kernels $\Omega_{j,1}$ and $\Omega_{j,2}$ (defined in section 5), to establish a correspondence between the solutions of the homogeneous equation $\mathcal{L}u = 0$ and the solutions of a semilinear equation.

Theorem 8.1 *Assume that $\mathcal{L}$ has no spectral values in $i\mathbb{R}^*$. Let $G(u, r, t)$ be a bounded function defined in $\mathbb{C} \times A(0, R_0)$, for some $R_0 > 0$, and let $\tau > a\nu$. Then, there are $R > 0$ and a one to one map between the space of continuous solutions of the equation $\mathcal{L}u = 0$ in $A(0, R)$ and the space of continuous solutions of the equation*

$$\mathcal{L}u = r^\tau |u| G(u, r, t) . \tag{8.1}$$

Furthermore, if v is a bounded solution of (8.1) in a cylinder $A(0, R)$, then v is continuous up to the circle S_0.

Proof. First note that since $\tau > a\nu$, then $r^\tau \in L^p_a(A(0, R))$, with a p satisfying $p > 2/(1 - \nu)$, and $||r^\tau||_{p,a} = C_1 R^{\delta_1}$ with C_1 and δ_1 positive. Consider the operator

$$\mathcal{P} : C^0(\overline{A(0, R)}) \longrightarrow C^0(\overline{A(0, R)}) ; \quad \mathcal{P}(f) = T_R(r^\tau |f| G(f, r, t))$$

where, as before, T_R denotes the integral operator on the cylinder $A(0, R)$. Note that since G is a bounded function, then $\mathcal{P}$ is well defined. It follows from the properties of T given in Theorem 7.3, from the boundedness of G, and from $r^\tau \in L^p_a$ that the operator $\mathcal{P}$ satisfies

$$\mathcal{L}(\mathcal{P}(f)) = r^\tau |f| G(f, r, t)$$

and

$$|\mathcal{P}(f)(r, t)| \leq C R^\delta ||r^\tau |f| G(f, r, t)||_{p,a} \leq C' R^{\delta'} ||f||_0 \qquad \forall f \in C^0(\overline{A(0, R)})$$

with C' and δ' positive. Hence, if $R > 0$ is small enough, $||\mathcal{P}|| \leq C' R^{\delta'} < 1$, and $\mathcal{P}$ is thus a contraction. Let $\mathcal{F} = (I - \mathcal{P})^{-1}$. The operator $\mathcal{F}$ realizes the one to one mapping between the space of continuous solutions of $\mathcal{L}u = 0$ and those of equation (8.1).

Now, we show that if v a bounded solution of (8.1) in a cylinder $A(0, R)$, then it is continuous. For a bounded solution v, the function $r^\tau |v| G(v, r, t)$ is bounded and is in $L^p_a(A(0, R))$. Consequently, $\mathcal{P}(v)$ is continuous up to the boundary S_0. The function $u = v - \mathcal{P}(v)$ is a bounded solution of $\mathcal{L}u = 0$ and so it is continuous up to S_0 (Remark 6.1). It follows that $v = u + \mathcal{P}(v)$ is also continuous up to S_0 $\square$

Let $\sigma_j = \sigma_j^+$ (or $\sigma_j = \sigma_j^-$) be a spectral value of $\mathcal{L}$ such that $\mathrm{Re}(\sigma_j) > 0$. Consider the Banach spaces $r^{\sigma_j} L^p_a(A(0, R))$ and $r^{\sigma_j} C^0_b(A(0, R))$ defined as

follows: $f \in r^{\sigma_j} L^p_a(A(0,R))$ if $\dfrac{f}{r^{\sigma_j}} \in L^p_a$ and $g \in r^{\sigma_j} C^0_b(A(0,R))$ if $\dfrac{g}{r^{\sigma_j}} \in C^0(A(0,R))$ and is bounded. The norms in these spaces are defined by

$$||f||_{p,a,\sigma_j} = ||\frac{f}{r^{\sigma_j}}||_{p,a} \quad \text{and} \quad ||g||_{0,\sigma_j} = ||\frac{g}{r^{\sigma_j}}||_0$$

Consider the operator T^j_R defined by

$$T^j_R F(r,t) = \frac{-1}{2\pi} \iint_{A(0,R)} \left[\Omega_{j,1}(z,\zeta) F(\zeta) + \overline{\Omega_{j,2}(z,\zeta)}\, \overline{F(\zeta)} \right] \frac{d\rho d\theta}{\rho} \tag{8.2}$$

where $\Omega_{j,1}$ and $\Omega_{j,2}$ denote the modified kernels defined in (5.11) and (5.12). Note that the estimates of Theorem 5.2 on the modified kernels imply that $T^j_R F$ is in $r^{\sigma_j} C^0_b(A(0,R))$ when F is in $r^{\sigma_j} L^p_a(A(0,R))$. Arguments similar to those used in the proof of Theorem 7.3 can be used to establish the following theorem.

Theorem 8.2 *For $p > 2$, the operator*

$$T^j_R : r^{\sigma_j} L^p_a(A(0,R)) \longrightarrow r^{\sigma_j} C^0_b(A(0,R))$$

satisfies

$$\mathcal{L} T^j_R F(r,t) = F(r,t) \quad \text{and} \quad ||T^j_R F||_{0,\sigma_j} \le C R^\delta ||F||_{p,a,\sigma_j} \tag{8.3}$$

where C and δ are positive constants.

Two functions u and v defined in the cylinder $A(0,R)$ are said to be similar if u/v is continuous in $A(0,R)$ and there exist positive constants C_1 and C_2 such that

$$C_1 \le \left| \frac{u(r,t)}{v(r,t)} \right| \le C_2, \qquad \forall (r,t) \in A(0,R)\,.$$

Theorem 8.3 *Let $\mathcal{L}$, τ, and G be as in Theorem* 8.1*. Then there exists $R > 0$ such that each continuous solution of $\mathcal{L}u = 0$ in $\overline{A(0,R)}$ is similar to a continuous solution of equation* (8.1).

Proof. Let u be a continuous solution of $\mathcal{L}u = 0$ on $\overline{A(0,R)}$. Let $\mu \ge 0$ be the order of u along S_0. If $\mu > 0$, then $\mu = \mathrm{Re}(\sigma^\pm_j)$ for some spectral value $\sigma^\pm_j$. Assume that $\sigma^-_j = \sigma^+_j = \sigma_j$. Then it follows from the Laurent series expansion that u is similar to a linear combination

$$u_0(r,t) = a^- w^-_j(r,t) + a^+ w^+_j(r,t)$$

of the basic solutions w^+_j and w^-_j in a cylinder $A(0,R_1)$ with $0 < R_1 < R$. Consider the operator

$$\mathcal{P}_j : r^{\sigma_j} C^0_b(A(0,R)) \longrightarrow r^{\sigma_j} C^0_b(A(0,R))$$

defined by $\mathcal{P}_j(f) = T_R^j(r^\tau|f|G(f,r,t))$ It follows from the hypotheses on τ, on G, and from Theorem 8.2 that

$$\mathcal{L}\mathcal{P}_j(f) = r^\tau|f|G(f,r,t) \quad \text{and} \quad ||\mathcal{P}_j(f)||_{0,\sigma_j} \leq KR^\delta||f||_{0,\sigma_j}$$

for some positive constants K and δ. In particular, $\mathcal{P}_j$ is a contraction, if R is small enough. Hence, the function $v = (I - \mathcal{P}_j)^{-1}(u)$ is a solution of equation (8.1) and it is also similar to u_0.

If $\mu = 0$ (then we are necessarily in the case where 0 is a spectral value), let $\mathcal{F}$ be the resolvent of $\mathcal{P}$ used in the proof of Theorem 8.1. Then $v = \mathcal{F}(u)$ is similar to u and solves (8.1) $\square$

A direct consequence of Theorem 8.1 and Theorem 6.6 is the following uniqueness result for the solutions of (8.1)

Theorem 8.4 *Suppose that the spectral values of $\mathcal{L}$ satisfy the following condition*

$$\mathrm{Re}(\sigma_j) = \mathrm{Re}(\sigma_k) \implies \sigma_j = \sigma_k, \qquad \forall \sigma_j, \sigma_k \in \mathrm{Spec}(\mathcal{L})\,.$$

Let u be a bounded solution of (8.1) in a cylinder $A(0,R)$. Suppose that there is a sequence of points $(r_k,t_k) \in A(0,R)$ such that $r_k \to 0$ and $u(r_k,t_k) = 0$ for every $k \in Z^+$. Then $u \equiv 0$

9 The second order equation: Reduction

This section deals with the second order operator $P = L\overline{L} + \mathrm{Re}(aL)$. We show that the equation $Pu = F$, with u and F real-valued, can be reduced to an equation of the form $\mathcal{L}u = G$.

As before, let $\lambda = a + ib \in \mathbb{R}^+ + i\mathbb{R}$, L be the vector field given by (1.1) and let $\beta(t) \in C^m(S^1,\mathbb{C})$, with $m \geq 2$, satisfies

$$\frac{1}{2\pi i}\int_0^{2\pi} \beta(t)dt = k \in \mathbb{Z}\,. \tag{9.1}$$

Consider the second order operator P defined as

$$P = L\overline{L} + \overline{\lambda}\beta(t)L + \lambda\overline{\beta(t)}\,\overline{L}\,. \tag{9.2}$$

Then,

$$Pu = |\lambda|^2u_{tt} - 2bru_{rt} + r^2u_{rr} + |\lambda|^2(\beta + \overline{\beta})u_t + [1 - i(\overline{\lambda}\,\beta - \lambda\overline{\beta})]ru_r.$$

Note that P is elliptic except along the circle $S_0 = \{0\} \times \mathbb{S}^1$, and that Pu is $\mathbb{R}$-valued when u is $\mathbb{R}$-valued.

With the operator P we associate a first order operator $\mathcal{L}$ and show that the equation $Pu = F$, with F real-valued, is equivalent to an equation of the form $\mathcal{L}w = G$. Let

$$B(t) = \exp \int_0^t \overline{\beta(s)} ds \,. \tag{9.3}$$

It follows from (9.1) that B is periodic with $\mathrm{Ind}(B) = -k$ and satisfies $LB = \lambda\overline{\beta}\, B$. Define the function $c(t)$ by

$$c(t) = -\overline{\lambda}\, \beta(t) \frac{B(t)}{\overline{B(t)}} = -\overline{\lambda}\, \beta(t) \exp\left[\int_0^t (\overline{\beta(s)} - \beta(s)) ds\right] . \tag{9.4}$$

Note that the function B satisfies also the equation

$$LB = -\overline{c(t)}\, \frac{B^2(t)}{\overline{B(t)}} \tag{9.5}$$

To each $\mathbb{R}$-valued function u, we associate the $\mathbb{C}$-valued function w defined by

$$w(r,t) = B(t)\overline{L}u(r,t) \,.$$

We will refer to w as the L-potential of u with respect to P. To the operator P we associate the first order operator $\mathcal{L}$ defined by

$$\mathcal{L}w = Lw - c(t)\overline{w} \,, \tag{9.6}$$

where $c(t)$ is given by (9.4). We have the following proposition.

Proposition 9.1 *Suppose that F is an $\mathbb{R}$-valued function and $u(r,t)$ is $\mathbb{R}$-valued and solves the equation*

$$Pu(r,t) = F(r,t). \tag{9.7}$$

in the cylinder $A(0,R)$. Then its L-potential w satisfies the equation

$$\mathcal{L}w(r,t) = B(t)F(r,t). \tag{9.8}$$

Conversely, if w is a solution of (9.8) in $A(0,R)$, then there is an $\mathbb{R}$-valued function u defined in $A(0,R)$ that solves (9.7) and whose L-potential is w. More precisely, the function u can be defined by

$$u(r,t) = \mathrm{Re} \int_{(r_0,t_0)}^{(r,t)} \frac{w(\rho,\theta)}{B(\theta)} \frac{d\zeta}{ia\zeta} \tag{9.9}$$

where $\zeta = \rho^\lambda \mathrm{e}^{i\theta}$ and the integration is taken over any simple curve in $A(0,R)$ that joins the fixed point (r_0,t_0) to the point (r,t)

Proof. Suppose that u is $\mathbb{R}$-valued and solves (9.7). By using (9.4) and (9.5), we see that its L-potential satisfies

$$\begin{aligned} Lw = L(B\overline{L}u) &= BL\overline{L}u + LB\overline{L}u = BF - \overline{\lambda}\beta BLu \\ &= BF - \overline{\lambda}\beta\frac{B}{\overline{B}}\overline{(B\overline{L}u)} = BF + c\overline{w} \end{aligned}$$

Thus w solves (9.8). Conversely, suppose that w solves (9.8). Let $(r_0, t_0) \in A(0, R)$ and consider the function $u(r, t)$ defined by (9.9). We need to verify that the integral is path independent. Let U be a relatively compact subset of $A(0, R)$ whose boundary consists of simple closed curves. It follows from the proof of Green's identity (1.8) and (9.5) that

$$\begin{aligned} \int_{\partial U} \frac{w(\zeta)}{B(\theta)}\frac{d\zeta}{ia\zeta} &= \iint_U L\left(\frac{w}{B}\right)\frac{d\overline{\zeta}d\zeta}{2a^2|\zeta|^2} = \iint_U \left[\frac{Lw}{B} - \frac{LB}{B^2}w\right]\frac{id\rho d\theta}{\rho} \\ &= \iint_U \left[F + \frac{c}{B}\overline{w} + \frac{\overline{c}}{\overline{B}}w\right]\frac{id\rho d\theta}{\rho} \end{aligned}$$

Since F is $\mathbb{R}$-valued, then the real part of the above integral is zero and the function u is well defined. That u satisfies (9.7) follows easily by computing the derivatives u_t and u_r from (9.9) to obtain $\overline{L}u = w/B$ and then using (9.8) to get (9.7) $\square$

10 The homogeneous equation $Pu = 0$

We use the reduction given in Proposition 9.1 to obtain properties of the solutions of the equation $Pu = 0$ from those of their L-potentials w. In particular, series representation for u in a cylinder is derived. Under an assumption on the spectrum of $\mathcal{L}$, we prove a maximum principle for the equation $Pu = 0$: The extreme values of u can occur only on the distinguished boundary $\partial_0 U$. It should be mentioned that many results in this section and the next are close to those obtained, in [10]. There, the operator in $\mathbb{R}^2$ has its principal part of the particular form $(x^2 + y^2)\Delta$, where Δ is the Laplacian. Such an operator, when written in polar coordinates has the form (9.2) with the vector field L having the invariant $\lambda = 1$.

10.1 Some properties

The following simple properties for the solutions u will be needed . We start by considering the possibility of the existence of radial solutions.

Proposition 10.1 *The equation $Pu = 0$ has radial solutions $u = u(r)$ if and only if the coefficient β has the form*

$$\beta(t) = \frac{\lambda}{a}p(t) - ik \tag{10.1}$$

where $k \in \mathbb{Z}$ and $p(t)$ is $\mathbb{R}$-valued and such that $\displaystyle\int_0^{2\pi} p(t)dt = 0$*. In this case, the radial solutions have the form*

$$u(r) = \begin{cases} C_1 \log r + C_2 & \text{if} \quad k = 0 \\ C_1 r^{2ak} + C_2 & \text{if} \quad k \neq 0 \end{cases} \tag{10.2}$$

where C_1, C_2 are arbitrary constants. The corresponding L-potentials are $w(r,t) = iC_1B(t)$, when $k = 0$ and $w(r,t) = 2iakC_1r^{2ak}B(t)$, when $k \neq 0$, where

$$B(t) = \mathrm{e}^{ikt} \exp\left(\frac{\overline{\lambda}}{a} \int_0^t p(s)ds\right)$$

Moreover, $w(r,t)$ is a basic solution of $\mathcal{L}$ with character $(2ak,\ k)$.

Proof. If $u = u(r)$ solves $Pu = 0$, then

$$r^2 u''(r) + (1 - i(\overline{\lambda}\,\beta(t) - \lambda\,\overline{\beta(t)}))ru'(r) = 0\,.$$

Hence, $i(\overline{\lambda}\,\beta(t) - \lambda\,\overline{\beta(t)})$ is a real constant. If we set $\beta(t) = p(t) + iq(t)$ with p and q real-valued, then $aq(t) - bp(t) = M$, with $M \in \mathbb{R}$ constant. It follows from hypothesis (9.1) that $M = -ak$ with $k \in \mathbb{Z}$ and that the average of p is zero. This gives $aq(t) = bp(t) - ak$ and consequently β has the form (10.1). For such a coefficient β, the radial solutions are easily obtained from the differential equation $\square$

Remark 10.1 Note that if $u = u(t)$ (independent on r) solves the equation $Pu = 0$, then u is necessarily constant.

The following lemma will be used in the proof of the next proposition.

Lemma 10.1 *Let $u(r,t)$ be a solution of $Pu = 0$ in the cylinder $A(0,R)$ and let $w = B\overline{L}u$ be its L-potential. If*

$$\mathrm{Re}\left[\frac{\lambda w(r,t)}{iaB(t)}\right] \equiv 0\,,$$

then u is constant.

Proof. Let $(r_0, 0)$ be a fixed point in the cylinder $A(0,R)$. Let u be as in the lemma and let $w = B\overline{L}u$. The function

$$v(r,t) = \mathrm{Re} \int_{\Gamma(r,t)} \frac{w(\zeta)}{B(\theta)} \frac{d\zeta}{ia\zeta}$$

where $\Gamma(r,t)$ is any piecewise smooth curve that joins the point $(r_0, 0)$ to the point (r,t), solves $Pv = 0$ (Proposition 9.1). We choose Γ as $\Gamma = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 = \{(r_0, st),\ 0 \leq s \leq 1\} \text{ and } \Gamma_2 = \{((1-s)r_0 + sr, t),\ 0 \leq s \leq 1\}.$$

With this choice of Γ and with the hypothesis of the lemma on the potential w, the integral over Γ_2 is 0 and the expression for v reduces to

$$v(r,t) = \frac{t}{a}\mathrm{Re}\int_0^1 \frac{w(r_0,st)}{B(st)}ds\,.$$

Hence, the function v depends only on the variable t and since it solves $Pv = 0$, then v is constant (Remark 10.1). Consequently, $w = B\overline{L}v = 0$. This means $\overline{L}u = 0$. Since u is $\mathbb{R}$-valued, then $Lu = 0$ and so u is constant $\square$

Proposition 10.2 *Suppose that $u \in C^0(\overline{A(0,R)})$ solves $Pu = 0$, then its L-potential w satisfies, $w \in C^0(A(0,R) \cup S_0)$ and $w(0,t) \equiv 0$. Moreover, u is constant along S_0.*

Proof. Since P is elliptic for $r \neq 0$, then we need only to verify the continuity of w up to S_0 and its vanishing there. As a solution of $\mathcal{L}w = 0$, the function w has a Laurent series expansion (Theorem 6.2)

$$w(r,t) = \sum_{j\in\mathbb{Z}} a_j^{\pm} w_j^{\pm}(r,t)$$

where $w_j^{\pm}$ are the basic solutions of $\mathcal{L}$. Let $\tau \in \mathbb{R}$ be the order of w along S_0 (that the order τ is finite is a consequence of the continuity of u up to S_0). We are going to show that $\tau > 0$. Let $w_1, \cdots, w_N$ be the collection of all basic solutions with order τ along S_0. That is, w_m is a basic solution with $\mathrm{Char}(w_m) = (\sigma_{j_m}, j_m)$ and such that the exponent satisfies $\mathrm{Re}(\sigma_{j_m}) = \tau$. We have then

$$w(r,t) = \sum_{k=1}^{N} a_k w_k(r,t) + o(r^\tau) = w_\tau(r,t) + o(r^\tau)$$

It follows from Lemma 10.1 that for each k, $\mathrm{Re}(\lambda w_k/iaB) \not\equiv 0$. Let $t_0 \in \mathbb{R}$ be such that

$$\mathrm{Re}\left(\frac{\lambda w_k(r,t_0)}{iaB(t_0)}\right) \neq 0, \qquad k = 1, \cdots, N.$$

Let $r_0 < R$ be fixed. By using integration over the segment from (r_0,t_0) to (r,t_0), we find

$$u(r,t_0) - u(r_0,t_0) = \int_0^1 \mathrm{Re}\left[\frac{\lambda w_\tau((1-s)r_0 + sr, t_0)}{iaB(t_0)}\right]\frac{(r-r_0)ds}{(1-s)r_0 + sr} + o(r^\tau).$$

Recall that each basic solution $w_1, \cdots, w_n$ has an exponent $\sigma_k = \tau + i\beta_k$ and so

$$\mathrm{Re}\int_0^1 \frac{(r-r_0)\lambda w_k((1-s)r_0 + sr, t_0)}{((1-s)r_0 + sr)iaB(t_0)}ds = \begin{cases} O(r^\tau) & \text{if } \tau \neq 0 \\ O(\log r) & \text{if } \tau = 0,\ \beta_k = 0 \\ O(r^{i\beta_k}) & \text{if } \tau = 0,\ \beta_k \neq 0 \end{cases}$$

From these estimates and the above integral, we deduce that in order for $u(r,t_0) - u(r_0,t_0)$ to have a limit as $r \to 0$, it is necessary that $\tau > 0$. For such τ, $w(0,t) = 0$, and $u(0,t)$ is constant. $\square$

As a consequence of the proof of Proposition 10.2, we have the following proposition.

Proposition 10.3 *Suppose that $\mathcal{L}$ has no spectral values on $i\mathbb{R}^*$. If $u \in L^\infty(A(0,R))$ solves $Pu = 0$, then u is continuous up to the boundary S_0 and it is constant on S_0.*

10.2 Main result about the homogeneous equation $Pu = 0$

We use the basic solutions of the associated operator $\mathcal{L}$ to construct 2π-periodic functions $q_j^\pm(t)$ and establish a series expansion of the continuous solutions u.

Let $\{\sigma_j^\pm\}_{j\in\mathbb{Z}}$ be the spectrum of the associated operator $\mathcal{L}$ and $w_j^\pm$ be the corresponding basic solutions. Recall that if $\sigma_j^\pm \in \mathbb{R}$, then $w_j^\pm = r^{\sigma_j^\pm} f_j^\pm(t)$ with $\mathrm{Ind}(f_j^\pm) = j$ and if $\sigma_j^+ \in \mathbb{C}\backslash\mathbb{R}$, then $\sigma_j^- = \sigma_j^+ = \sigma_j$ and

$$w_j^+(r,t) = r^{\sigma_j}\phi_j(t) + \overline{r^{\sigma_j}\psi_j(t)}\,, \qquad w_j^-(r,t) = i\left[r^{\sigma_j}\phi_j(t) - \overline{r^{\sigma_j}\psi_j(t)}\right]$$

with $|\phi_j| > |\psi_j|$ and $\mathrm{Ind}(\phi_j) = j$. Define the functions $q_j^\pm(t)$ as follows. For $\sigma_j^\pm \in \mathbb{R}^*$,

$$q_j^\pm(t) = \frac{\lambda}{ia\sigma_j^\pm}\frac{f_j^\pm(t)}{B(t)} \tag{10.3}$$

and for $\sigma_j \in \mathbb{C}\backslash\mathbb{R}$,

$$q_j^+(t) = \frac{1}{ia\sigma_j}\left(\frac{\lambda\phi_j(t)}{B(t)} - \frac{\overline{\lambda}\,\overline{\psi_j(t)}}{\overline{B(t)}}\right), \quad q_j^-(t) = \frac{1}{a\sigma_j}\left(\frac{\lambda\phi_j(t)}{B(t)} + \frac{\overline{\lambda}\,\overline{\psi_j(t)}}{\overline{B(t)}}\right). \tag{10.4}$$

It follows from Theorem 4.1 that the asymptotic behaviors of $q_j^\pm$ are

$$q_j^+(t) = \frac{\mathrm{e}^{ijt}}{iajB(t)} + O(j^{-2}) \quad \text{and} \quad q_j^-(t) = \frac{\mathrm{e}^{ijt}}{ajB(t)} + O(j^{-2}).$$

We have the following representation theorem

Theorem 10.1 *If $u \in C^0(\overline{A(0,R)})$ is a solution of $Pu = 0$, then u is constant on S_0 and it has the series expansion*

$$u(r,t) = u_0 + \sum_{\mathrm{Re}(\sigma_j^\pm)>0} u_j^\pm \mathrm{Re}\left[r^{\sigma_j^\pm} q_j^\pm(t)\right] \tag{10.5}$$

where the functions $q_j^{\pm}$ *are defined in* (10.3) *and* (10.4), *and where* $u_j^{\pm} \in \mathbb{R}$.

Proof. It follows from Proposition 10.2 that u is constant on S_0. Hence, by using integration over the segment from $(0,t)$ to (r,t), we obtain

$$u(r,t) - u(0,0) = u(r,t) - u(0,t) = \mathrm{Re}\int_0^1 \frac{\lambda w(sr,t)}{iaB(t)}\frac{ds}{s}$$

where w is the L-potential of u. The function w, being a solution of $\mathcal{L}w = 0$, has a series expansion

$$w(r,t) = \sum_{\mathrm{Re}(\sigma_j^{\pm})>0} c_j^{\pm} w_j^{\pm}(r,t).$$

For the function u we have then

$$u(r,t) = u(0,0) + \sum_{\mathrm{Re}(\sigma_j^{\pm})>0} c_j^{\pm}\mathrm{Re}\int_0^1 \frac{\lambda w_j^{\pm}(sr,t)}{iaB(t)}\frac{ds}{s}\,.$$

Now for $\sigma_j^{\pm} \in \mathbb{R}$, we have $w_j^{\pm}(r,t) = r^{\sigma_j^{\pm}} f_j^{\pm}(t)$ and

$$\int_0^1 \frac{\lambda w_j^{\pm}(sr,t)}{iaB(t)}\frac{ds}{s} = r^{\sigma_j^{\pm}}\frac{\lambda}{ia\sigma_j^{\pm}}\frac{f_j^{\pm}(t)}{B(t)} = r^{\sigma_j^{\pm}} q_j^{\pm}(t)\,.$$

For $\sigma_j^+ = \sigma_j^- = \sigma_j \in \mathbb{C}\backslash\mathbb{R}$, we have

$$\begin{aligned}\int_0^1 \frac{\lambda w_j^{+}(sr,t)}{iaB(t)}\frac{ds}{s} &= \int_0^1 \left[(rs)^{\sigma_j}\frac{\lambda\phi_j(t)}{iaB(t)} + (rs)^{\overline{\sigma_j}}\frac{\lambda\overline{\psi_j(t)}}{iaB(t)}\right]\frac{ds}{s}\\ &= \frac{r^{\sigma_j}}{\sigma_j}\frac{\lambda\phi_j(t)}{iaB(t)} + \frac{r^{\overline{\sigma_j}}}{\overline{\sigma_j}}\frac{\lambda\overline{\psi_j(t)}}{iaB(t)}.\end{aligned}$$

From this and (10.4), we get

$$\mathrm{Re}\int_0^1 \frac{\lambda w_j^{+}(sr,t)}{iaB(t)}\frac{ds}{s} = \mathrm{Re}\left(r^{\sigma_j} q_j^{+}(t)\right)$$

A similar relation holds for the integral of w_j^- and the series expansion (10.5) follows $\square$

Remark 10.2 A consequence of this theorem and of the asymptotic expansion of the spectral values σ_j, given in Theorem 4.1, imply that the number λ is an invariant for the operator P in the following sense: Suppose that

$$P_1 = L_1\overline{L_1} + \overline{\lambda_1}\beta_1(t)L_1 + \lambda_1\overline{\beta_1(t)}\,\overline{L_1}$$

is generated by a vector field L_1 with invariant $\lambda_1 = a_1 + ib_1 \in \mathbb{R}^+ + i\mathbb{R}$ and such that for every $k \in \mathbb{Z}^+$, there is a diffeomorphism, Φ^k, in a neighborhood of the circle S_0 such that that $\Phi_*^k P$ is a multiple of P_1, then $\lambda = \lambda_1$

10.3 A maximum principle

We use the series representation of Theorem 10.1 to obtain a maximum principle when the spectrum satisfies a certain condition.

Recall that the function $B(t)$ satisfies $\mathrm{Ind}(B) = -k$, where $k \in \mathbb{Z}$ is defined by (9.1). We will say that the operator P satisfies hypothesis $\mathcal{H}$ if the spectrum of $\mathcal{L}$ satisfies the following conditions.

$\mathcal{H}_1 :\ \mathrm{Re}(\sigma_j^{\pm}) \leq 0 \implies j \leq -k.$

$\mathcal{H}_2 :\ \mathrm{Re}(\sigma_j^{\pm}) = \mathrm{Re}(\sigma_m^{\pm}) \implies \sigma_j^{\pm} = \sigma_m^{\pm}$

Thus P satisfies $\mathcal{H}$ means that the projection of $\mathrm{Spec}(\mathcal{L})$ into $\mathbb{R}$ is injective and that the basic solutions w of $\mathcal{L}$ with positive orders have winding numbers $\mathrm{Ind}(w) > k$.

Theorem 10.2 *Suppose that the operator P satisfies $\mathcal{H}$. Let $U \subset \mathbb{R}^+ \times \mathbb{S}^1$ be open, bounded, and such that $A(0, R) \subset U$ for some $R > 0$. If $u \in C^0(\overline{U})$ satisfies $Pu = 0$, then the value of u on S_0 is not an extreme value of u. Thus the maximum and minimum of u occur on $\partial U \backslash S_0$.*

Proof. Let $\tau > 0$ be the order along S_0 of the L-potential of u. It follows from Theorem 10.1 that

$$u(r, t) - u(0, 0) = \sum_{\mathrm{Re}(\sigma_j^{\pm}) = \tau} c_j^{\pm} \mathrm{Re}\left(r^{\sigma_j^{\pm}} q_j^{\pm}(t)\right) + o(r^{\tau}). \tag{10.6}$$

We consider two cases depending on whether τ is a spectral value of $\mathcal{L}$ or τ is only the real part of a spectral value. Note that it follows from $\mathcal{H}_2$ that the sum in (10.6) consists of either one term or two terms. It has one term, if τ is a spectral value with multiplicity one. It has two terms if τ is a spectral value with multiplicity two or if τ is not a spectral value.

If τ is spectral value (say with multiplicity 2), then the corresponding basic solutions have the form $r^{\tau} f_j^{\pm}(t)$ with winding number $j > -k$ (by condition $\mathcal{H}_1$). After replacing, in (10.6), the functions $q_j^{\pm}$ by their expressions given in (10.3), we find that

$$u(r, t) - u(0, 0) = r^{\tau} \mathrm{Re}\left(\frac{\lambda}{ia\tau} \frac{c_j^+ f_j^+(t) + c_j^- f_j^-(t)}{B(t)}\right) + o(r^{\tau})$$

Recall that the functions f_j^+ and f_j^- are $\mathbb{R}$-independent solutions of the differential equation (2.3). Thus, $c_j^+ f_j^+ + c_j^- f_j^-$ has winding number j and consequently

$$\mathrm{Ind}\left(\frac{\lambda}{ia\tau} \frac{c_j^+ f_j^+(t) + c_j^- f_j^-(t)}{B(t)}\right) = j + k > 0$$

(we have used the fact that $\mathrm{Ind}(B) = -k$). Since the winding number is positive, the real part changes sign. This implies that $u(r,t)-u(0,0)$ changes sign (for r small) and $u(0,0)$ is not an extreme value. The proof for the case when τ is a spectral value with multiplicity one is similar.

If τ is not a spectral value, then there is a unique spectral value $\sigma = \tau+i\mu$ with $\mu \neq 0$. The corresponding basic solutions $w_j^{\pm}$ have winding number $j > -k$. After substituting, in (10.6), the functions $q_j^{\pm}$ by their expressions given in (10.4) we find

$$u(r,t) - u(0,0) = r^{\tau}\mathrm{Re}\left[\frac{r^{i\mu}}{ia\sigma}\left(D\lambda\frac{\phi_j(t)}{B(t)} - \overline{D\lambda}\frac{\psi_j(t)}{\overline{B(t)}}\right)\right] + o(r^{\tau})\,,$$

where $D = c^+ + ic^-$ and where ϕ_j and ψ_j are the components of the basic solutions. We have $|\phi_j| > |\psi_j|$ and $\mathrm{Ind}(\phi_j) = j$. The same argument as before shows that $u(r,t)-u(0,0)$ changes sign, as the real part of a function with winding number $j+k > 0$ $\square$

Remark 10.3 If the condition $\mathcal{H}$ is not satisfied, then equation $Pu = 0$ might have solutions with extreme values on S_0. Consider for example the case in which the function $\beta(t)$ is given by (10.1) with $k = 1$ (In this example we have $\mathrm{Ind}(B) = 1$). The operator P does not satisfy $\mathcal{H}_1$. Indeed, $2a$ is a spectral value, corresponding to the basic solution $r^{2a}B(t)$, with winding number 1. The corresponding basic solution r^{2a} has minimum value 0 and it is attained on S_0.

11 The nonhomogeneous equation $Pu = F$

We construct here integral operators for the equation $Pu = F$. A similarity principle between the solutions of $Pu = 0$ and those of a semilinear equation is then obtained through these operators.

Let $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ be the functions given by (7.7). Define the function $S(z,\zeta)$ by

$$S(z,\zeta) = \mathrm{Re}\left[\frac{-\lambda}{2\pi ai B(t)}\int_0^1\left(\widehat{\Omega}_1(sr,t,\zeta)B(\theta) + \overline{\widehat{\Omega}_2(sr,t,\zeta)B(\theta)}\right)\frac{ds}{s}\right] \tag{11.1}$$

and the integral operator $\mathbb{K}$ by

$$\mathbb{K}F(r,t) = \iint_{A(0,R)} S(r,t,\rho,\theta)F(\rho,\theta)\frac{d\rho d\theta}{\rho} \tag{11.2}$$

We have the following theorem.

Theorem 11.1 *If $p > 2$ and $R > 0$, then there exist positive constants C and δ such that $\mathbb{K} : L_a^p(A(0,R)) \longrightarrow C^0(\overline{A(0,R)})$ has the following*

properties

$$P(\mathbb{K}F) = F, \quad \mathbb{K}F(0,t) = 0, \quad \text{and} \quad |\mathbb{K}F(r,t)| \leq CR^{\delta}||F||_{p,a}\,.$$

Proof. For an $\mathbb{R}$-valued function $F \in L^p_a(A(0,R))$, with $p > 2$, consider

$$\widehat{T}_R(B(t)F(z)) = \frac{-1}{2\pi}\iint_{A(0,R)}\Big(\widehat{\Omega}_1(z,\zeta)B(\theta) + \overline{\widehat{\Omega}_2(z,\zeta)B(\theta)}\Big)\, F(\zeta)\frac{d\rho d\theta}{\rho} \tag{11.3}$$

We know, from Theorem 7.4, that $\widehat{T}_R(BF) \in C^0(\overline{A(0,R)})$ satisfies

$$\mathcal{L}\widehat{T}_R(BF) = BF, \quad \widehat{T}_R(BF)(0,t) = 0, \quad \text{and}$$
$$|\widehat{T}_R(BF)(r,t)| \leq CR^{\delta}||BF||_{p,a} \leq C||B||_0R^{\delta}||F||_{p,a} \quad \forall F \in L^p_a(A(0,R)).$$

for some positive constants C_1 and δ. Furthermore, it follows from (11.1), (11.2), and (11.3) that

$$\mathbb{K}(F)(r,t) = \mathrm{Re}\left(\frac{\lambda}{ia}\int_0^1 \frac{\widehat{T}_R(BF)(sr,t)}{B(t)}\,\frac{ds}{s}\right). \tag{11.4}$$

Then, from Proposition 9.1, we conclude that $\widehat{T}_R(BF)$ is the L-potential of $\mathbb{K}(F)$. The conclusion of the theorem follows from (11.4) and from the properties of $\widehat{T}_R$ $\square$

To established a similarity principle between the solutions of $Pu = 0$ and those of an associated semilinear equation, we need to use the modified kernels of section 5. For $j \in \mathbb{Z}$, let $\Omega^{\pm}_{j,1}$ and $\Omega^{\pm}_{j,2}$ be the kernels given by (5.11) and (5.12). Define $S^{\pm}_j$ by

$$S^{\pm}_j(z,\zeta) = \mathrm{Re}\left[\frac{-\lambda}{2\pi aiB(t)}\int_0^1\Big(\Omega^{\pm}_{j,1}(sr,t,\zeta)B(\theta) + \overline{\Omega^{\pm}_{j,2}(sr,t,\zeta)B(\theta)}\Big)\,\frac{ds}{s}\right] \tag{11.5}$$

and the operator $\mathbb{K}^{\pm}_j$ by

$$\mathbb{K}^{\pm}_jF(r,t) = \iint_{A(0,R)} S^{\pm}_j(r,t,\rho,\theta)F(\rho,\theta)\frac{d\rho d\theta}{\rho}. \tag{11.6}$$

The operators $T^{\pm}_j$, defined in (8.2), and $\mathbb{K}^{\pm}_j$ are related by

$$\mathbb{K}^{\pm}_jF(r,t) = \mathrm{Re}\left(\frac{\lambda}{ia}\int_0^1 \frac{T^{\pm}_j(BF)(sr,t)}{B(t)}\,\frac{ds}{s}\right) \tag{11.7}$$

The operator $\mathbb{K}^{\pm}_j$ acts on the Banach space $r^{\sigma^{\pm}_j}L^p_a(A(0,R))$, defined in Section 8, and produces continuous functions that vanish along S_0. More precisely, define the Banach space $r^{\sigma^{\pm}_j}\mathcal{E}(A(0,R))$ to be the set of functions $v(r,t)$

that are in $C^1(A(0,R))$ such that $\dfrac{v}{r^{\sigma_j^\pm}}$ and $\dfrac{Lv}{r^{\sigma_j^\pm}}$ are bounded functions in $A(0,R)$. The norm of v is

$$\|v\|_{r^{\sigma_j^\pm}\mathcal{E}} = \left\| \frac{v}{r^{\sigma_j^\pm}} \right\|_0 + \left\| \frac{Lv}{r^{\sigma_j^\pm}} \right\|_0 .$$

The next theorem can be proved by using Theorem 8.2 and arguments similar to those used in the proof of Theorem 11.1.

Theorem 11.2 *The operator*

$$\mathbb{K}_j^\pm : r^{\sigma_j^\pm} L_a^p(A(0,R)) \longrightarrow r^{\sigma_j^\pm}\mathcal{E}(A(0,R))$$

satisfies $P\mathbb{K}_j^\pm F = F$ *and*

$$\|\mathbb{K}_j^\pm F\|_{r^{\sigma_j^\pm}\mathcal{E}} \le CR^\delta \|F\|_{p,a,\sigma_j^\pm}$$

for some positive constants C *and* δ*.*

Let $f_0(r,t)$, $f_1(r,t)$, and $f_2(r,t)$ be bounded functions in $A(0,R)$ and let $g_1(r,t,u,w)$ and $g_2(r,t,u,w)$ be bounded functions in $A(0,R)\times\mathbb{R}\times\mathbb{C}$. Define the function H by

$$H(r,t,u,w) = uf_0 + wf_1 + \overline{w}f_2 + |u|^{1+\alpha}g_1 + |w|^{1+\alpha}g_2 \tag{11.8}$$

with $\alpha > 0$. For $\epsilon > 0$, consider the semilinear equation

$$Pu = r^\epsilon \mathrm{Re}\left(H(r,t,u,Lu)\right) . \tag{11.9}$$

We have the following similarity result between the solutions of (11.9) and those of the equation $Pu = 0$.

Theorem 11.3 *For a given function* H *defined by* (11.8)*, there exists* $R > 0$ *such that, for every* $u \in C^0(\overline{A(0,R)})$ *satisfying* $Pu = 0$ *and* $u = 0$ *on* S_0*, there exists a function* $m \in C^0(A(0,R))$ *satisfying*

$$C_1 \le m(r,t) \le C_2 \qquad \forall (r,t) \in A(0,R)$$

with C_1 *and* C_2 *positive constants, such that the function* $v = mu$ *solves equation* (11.9)*.*

Proof. Let u be a solution of $Pu = 0$ with order $\tau > 0$ along S_0. Then there is $\sigma_j^\pm \in \mathrm{Spec}(\mathcal{L})$ such that $\tau = \mathrm{Re}(\sigma_j^\pm)$. Hence, $u \in r^{\sigma_j^\pm}\mathcal{E}(A(0,R_0))$ for some $R_0 > 0$. Consider the operator

$$\mathcal{Q} :\ r^{\sigma_j^\pm}\mathcal{E}(A(0,R_0)) \longrightarrow r^{\sigma_j^\pm}\mathcal{E}(A(0,R_0))$$

given by $\mathcal{Q}v = \mathbb{K}_j^{\pm}(r^{\epsilon}\mathrm{Re}(H(r,t,v,Lv)))$. It follows from (11.8) that the function $r^{\epsilon}\mathrm{Re}(H(r,t,v,Lv))$ is in the space $r^{\sigma_j^{\pm}}L_a^p$. Now, Theorems 11.2, 8.2, and relation (11.7), imply that

$$\begin{aligned} P\mathcal{Q}v(r,t) &= r^{\epsilon}\mathrm{Re}(H(r,t,v,Lv)), \\ L\mathcal{Q}v(r,t) &= T_j^{\pm}\left[B(t)r^{\epsilon}\mathrm{Re}(H(r,t,v,Lv))\right]. \end{aligned}$$

Consequently, $\|\mathcal{Q}v\|_{r^{\sigma_j^{\pm}}\mathcal{E}} \le CR_0^{\delta}\|v\|_{r^{\sigma_j^{\pm}}\mathcal{E}}$. If R_0 is small enough, we have $\|\mathcal{Q}\| < 1$, and we can define the resolvent $\mathcal{F} = (I - \mathcal{Q})^{-1}$. It is easily checked that for the solution u of $Pu = 0$ as above, the function $v = \mathcal{F}(u)$ solves equation (11.9) and $m = u/v$ is bounded away from 0 and ∞ $\square$

12 Normalization of a Class of Second Order Equations with a Singularity

This section deals with the normalization of a class of second order operators $\mathbb{D}$ in $\mathbb{R}^2$ whose coefficients vanish at a point. To such an operator, a complex number $\lambda = a + ib \in \mathbb{R}^+ + i\mathbb{R}$ is invariantly associated. It is then shown that the operator $\mathbb{D}$ is conjugate, in a punctured neighborhood of the singularity, to a unique operator $\mathbb{P}$ given by (9.2). The properties of the solutions of the equations corresponding to $\mathbb{D}$ are, thus, inherited from the solutions of the equations for $\mathbb{P}$ studied in sections 10 and 11.

Let $\mathbb{D}$ be the second order operator given in a neighborhood of $0 \in \mathbb{R}^2$ by

$$\mathbb{D}u = a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y \tag{12.1}$$

where the coefficients $a_{11}, \cdots, a_2$ are C^{∞}, real-valued functions vanishing at 0, with a_{11} nonnegative, and

$$C_1 \le \frac{a_{11}(x,y)a_{22}(x,y) - a_{12}(x,y)^2}{(x^2+y^2)^2} \le C_2 \tag{12.2}$$

for some positive constants $C_1 < C_2$. It follows in particular that a_{11} and a_{22} vanish to second order at 0. Let A and B be the functions defined for $(x,y) \ne 0$ by

$$\begin{aligned} A(x,y) &= \frac{(x^2+y^2)\sqrt{a_{11}a_{22}-a_{12}^2}}{a_{11}y^2 - 2a_{12}xy + a_{22}x^2} \\ B(x,y) &= \frac{(a_{22}-a_{11})xy + a_{12}(x^2-y^2)}{a_{11}y^2 - 2a_{12}xy + a_{22}x^2} \end{aligned} \tag{12.3}$$

Note that it follows from (12.2) that these functions are bounded and A is positive. Let

$$\mu = \frac{1}{2\pi}\lim_{\rho \to 0^+}\int_{C_\rho} \frac{A(x,y) - iB(x,y)}{x^2+y^2}\,(xdy - ydx)\,, \tag{12.4}$$

where C_ρ denotes the circle with radius ρ and center 0. We will prove that $\mu \in \mathbb{R}^+ + i\mathbb{R}$ is well defined and it is an invariant for the operator $\mathbb{D}$.

We will be using the following normalization theorem for a class of vector fields in a neighborhood of a characteristic curve.

Theorem 12.1 *Let X be a C^∞ complex vector field in $\mathbb{R}^2$ satisfying the following conditions in a neighborhood of a smooth, simple, closed curve Σ:*

(i) $X_p \wedge \overline{X}_p \neq 0$ for every $p \notin \Sigma$;

(ii) $X_p \wedge \overline{X}_p$ vanishes to first order for $p \in \Sigma$; and

(iii) X is tangent to Σ.

Then there exist an open tubular neighborhood U of Σ, a positive number R, a unique complex number $\lambda \in \mathbb{R}^+ + i\mathbb{R}$, and a diffeomorphism

$$\Phi : U \longrightarrow (-R,\ R) \times \mathbb{S}^1$$

such that

$$\Phi_* X = m(r,t)\left[\lambda \frac{\partial}{\partial t} - ir\frac{\partial}{\partial r}\right]$$

where $m(r,t)$ is a nonvanishing function. Moreover, when $\lambda \notin \mathbb{Q}$, then for any given $k \in \mathbb{Z}^+$, the diffeomorphism Φ and the function m can be taken to be of class C^k.

This normalization Theorem was proved in [7] when $\lambda \in \mathbb{C}\backslash\mathbb{R}$. When $\lambda \in \mathbb{R}$, only a C^1-diffeomorphism Φ is achieved in [7]. A generalization is obtained by Cordaro and Gong in [4] to include C^k-smoothness of Φ when $\lambda \in \mathbb{R}\backslash\mathbb{Q}$. It is also proved in [4], that, in general, a C^∞-normalization cannot be achieved.

We will be using polar coordinates $x = \rho\cos\theta$, $y = \rho\sin\theta$ and we will denote this change of coordinates by Ψ. Thus,

$$\Psi : \mathbb{R}^2\backslash 0 \longrightarrow \mathbb{R}^+ \times \mathbb{S}^1 , \qquad \Psi(x,y) = (\rho,\theta) .$$

Theorem 12.2 *Let $\mathbb{D}$ be the second order operator given by* (12.1) *whose coefficients vanish at* 0 *and satisfy condition* (12.2)*. Then there is a neighborhood U of the circle $\{0\} \times \mathbb{S}^1$ in $[0,\ \infty) \times \mathbb{S}^1$, a positive number R, a diffeomorphism*

$$\Phi : U \longrightarrow [0,\ R) \times \mathbb{S}^1$$

sending $\{0\} \times \mathbb{S}^1$ onto itself, such that

$$(\Phi \circ \Psi)_* \mathbb{D} = m(r,t)\left[L\overline{L} + \mathrm{Re}(\beta(r,t)L)\right] \tag{12.5}$$

where m, β are differentiable functions with $m(r,t) \neq 0$ for every (r,t) and

$$L = \lambda\frac{\partial}{\partial t} - ir\frac{\partial}{\partial r}$$

with $\lambda = \dfrac{1}{\mu}$ *and* μ *given by* (12.4). *Moreover, if the invariant* $\mu \notin \mathbb{Q}$*, then for every* $k \in \mathbb{Z}^+$*, the diffeomorphism* Φ*, and the functions* m*, and* β *can be chosen to be of class* C^k*.*

Proof. We start by rewriting $\mathbb{D}$ in polar coordinates:

$$\mathbb{D}u = Pu_{\theta\theta} + 2Nu_{\rho\theta} + Mu_{\rho\rho} + Qu_\rho + Tu_\theta \tag{12.6}$$

where

$$\begin{aligned}
P &= \frac{1}{\rho^2}\left[a_{11}\sin^2\theta - 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta\right] \\
N &= \frac{1}{\rho}\left[-a_{11}\sin\theta\cos\theta + a_{12}(\cos^2\theta - \sin^2\theta) + a_{22}\cos\theta\sin\theta\right] \\
M &= a_{11}\cos^2\theta + 2a_{12}\sin\theta\cos\theta + a_{22}\sin^2\theta \\
Q &= \frac{1}{\rho}\left[a_{11}\sin^2\theta - 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta\right] + a_1\cos\theta + a_2\sin\theta \\
T &= \frac{1}{\rho^2}\left[a_{11}\sin\theta\cos\theta + a_{12}(\sin^2\theta - \cos^2\theta) - a_{22}\sin\theta\cos\theta\right] - \frac{1}{\rho}(a_1\sin\theta + a_2\cos\theta)
\end{aligned}$$

Condition (12.2) implies that there is a constant $C_0 >$ such that

$$M(\rho,\theta) \geq C_0\rho^2 \quad \text{and} \quad P(\rho,\theta) \geq C_0 \qquad \forall(\rho,\theta).$$

We define the following C^∞ functions (of (ρ,θ))

$$N_1 = \frac{N}{\rho P}, \quad M_1 = \frac{M}{\rho^2 P}, \quad Q_1 = \frac{Q}{\rho P}, \quad T_1 = \frac{T}{P}.$$

In terms of these function, (12.2) takes the form

$$M_1(\rho,\theta) - N_1^2(\rho,\theta) \;\geq\; C_2, \qquad \forall(\rho,\theta) \in [0,\ R_1] \times \mathbb{S}^1\,, \tag{12.7}$$

and (12.6) becomes

$$\frac{\mathbb{D}u}{P} = u_{\theta\theta} + 2\rho N_1 u_{\rho\theta} + \rho^2 M_1 u_{\rho\rho} + \rho Q_1 u_\rho + T_1 u_\theta \tag{12.8}$$

Let X be the C^∞ complex vector field defined by

$$X = \frac{\partial}{\partial\theta} - \rho g(\rho,\theta)\frac{\partial}{\partial\rho} \tag{12.9}$$

with $g = N_1 + i\sqrt{M_1 - N_1^2}$. Although we will use X for $\rho \geq 0$, the vector field X is defined in a neighborhood of $\{0\} \times \mathbb{S}^1$ in $\mathbb{R} \times \mathbb{S}^1$. By using X and its complex conjugate $\overline{X}$, we find that

$$X\overline{X}u = u_{\theta\theta} + 2\rho N_1 u_{\theta\rho} + \rho^2 M_1 u_{\rho\rho} + \rho f u_\rho \tag{12.10}$$

where

$$f = \frac{X(\rho\overline{g})}{\rho} = -|g|^2 + X(\overline{g}).$$

We also have

$$\rho u_\rho = \frac{Xu - \overline{X}u}{r - \overline{g}} \quad \text{and} \quad u_\theta = \frac{g\overline{X}u - \overline{g}Xu}{g - \overline{g}}. \tag{12.11}$$

It follows from (12.8), (12.10) and (12.11) that

$$\frac{\mathbb{D}u}{P} = X\overline{X}u - \frac{f - Q - 1 + \overline{g}T_1}{g - \overline{g}}\, Xu + \frac{f - Q - 1 + gT_1}{g - \overline{g}}\, \overline{X}u \tag{12.12}$$

Since the coefficients of $\mathbb{D}$ and the function u are $\mathbb{R}$-valued, then the right hand side of (12.12) is real valued and can be written as

$$2\frac{\mathbb{D}u}{P} = X\overline{X}u + \overline{X}\, Xu + B(\rho,\theta)Xu + \overline{B(\rho,\theta)}\,\overline{X}u \tag{12.13}$$

with

$$B(\rho,\theta) = -\frac{f + \overline{f} - 2Q_1 + 2\overline{g}T_1}{g - \overline{g}}.$$

Now, for the vector field X, we have

$$X \wedge \overline{X} = \rho(\overline{g} - g)\frac{\partial}{\partial\theta} \wedge \frac{\partial}{\partial\rho} = -2i\rho\sqrt{M_1 - N_1^2}\,\frac{\partial}{\partial\theta} \wedge \frac{\partial}{\partial\rho},$$

and so X satisfies the conditions of Theorem 12.1 and therefore it can be normalized. In our setting, the invariant λ is given by (see [7]) $\lambda = 1/\widetilde{\mu}$ where

$$\widetilde{\mu} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sqrt{M_1(0,\theta) - N_1^2(0,\theta)} - iN_1(0,\theta)\right) d\theta \,= \mu,$$

and where μ is given by (12.4). Hence, there is a diffeomorphism Φ defined in a neighborhood of $\rho = 0$ in $\mathbb{R}\times\mathbb{S}^1$ onto a cylinder $(-R,\ R)\times\mathbb{S}^1$ such that $\Phi_*X = m(r,t)L$ with L as in the Theorem and m a nonvanishing function. Finally, it follows from this normalization of X that, in the (r,t) coordinates, expression (12.13) becomes

$$2\frac{\mathbb{D}u}{P} = 2|m|^2L\overline{L}u + (mB + \overline{m}\,\overline{L}m)\,Lu + (\overline{m}\,\overline{B} + mL\overline{m})\,\overline{L}u\,. \tag{12.14}$$

This completes the proof of the theorem $\square$

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